Chapter 11. Fourier Analysis

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Outline

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2. Overview of Chapter 11
3. Sec. 11.1 Fourier Series
4. Sec. 11.2 Arbitrary Period. Even and Odd Functions. Half-Range Expansions
5. Sec. 11.3 Forced Oscillations
6. Sec. 11.4 Approximation by Trigonometric Polynomials
7. Sec. 11.5 Sturm–Liouville Problems. Orthogonal Functions
8. Sec. 11.6 Orthogonal Series. Generalized Fourier Series
9. Sec. 11.7 Fourier Integral
10. Sec. 11.8 Fourier Cosine and Sine Transforms
11. Sec. 11.9 Fourier Transform. Discrete and Fast Fourier Transforms
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Function

Practice 1 (Shifting, Scaling, Reversal)

$x(t)$ is plotted in Fig. 4. Based on $x(t)$, plot the following functions.

1. $x(t + 1)$;
2. $x(-t + 1)$;
3. $x\left(\frac{3}{2}t\right)$
4. $x\left(\frac{3}{2}t + 1\right)$

Figure 1: $x(t)$
Function

Answer

Figure 2: $x(t + 1)$

Figure 3: $x(-t + 1)$
Function

Answer

Figure 4: $x\left(\frac{3}{2}t\right)$

Figure 5: $x\left(\frac{3}{2}t + 1\right)$
Trigonometric Function

Practice 2 (Trigonometric Identities)

\[
\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B
\]
\[
\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B.
\]

Review and Prove the following relationships using the above Angle sum and difference identities for Sine/Cosine.

1. \[
\cos A \cos B = \frac{1}{2} [\cos(A - B) + \cos(A + B)]
\]
2. \[
\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)]
\]
3. \[
\sin A \cos B = \frac{1}{2} [\sin(A + B) + \sin(A - B)]
\]
4. \[
\cos A \sin B = \frac{1}{2} [\sin(A + B) - \sin(A - B)]
\]
5. \[
\cos^2 A + \sin^2 A = 1
\]
6. \[
\sin 2A = 2 \sin A \cos A
\]
7. \[
\cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A
\]
8. \[
\cos^2 A = \frac{1 + \cos 2A}{2}; \quad \sin^2 A = \frac{1 - \cos 2A}{2}
\]
Trigonometric Integrals

Practice 3 (Trigonometric Integrals)

1. \( v(t) = V \sin \omega t \). \( T_p = \frac{2\pi}{\omega} \) is the period of \( v(t) \). \( V \) is a constant. Find 
   \[ \int_0^{T_p} v(t) \, dt = ? \]

2. \( g(t) = \sin \omega t \cos \omega t \). \( T_p = \frac{2\pi}{\omega} \) is the period of \( g(t) \). Find 
   \[ \int_0^{T_p} g(t) \, dt = ? \]

3. \( v(t) = V \sin \omega t \). \( T_p = \frac{2\pi}{\omega} \) is the period of \( v(t) \). Find 
   \[ I \equiv \int_0^{T_p} v^2(t) \, dt = ? \]
Integration by Parts

- Integration by Parts: \[ \int u \, dv = uv - \int v \, du \]

Practice 4 (Trigonometric Integrals using Integration by Parts)

1. \[ \int x^2 \cos nx \, dx = ? \]
2. \[ \int_{-\pi}^{\pi} x^2 \cos x \, dx = ? \]
3. \[ \int x^2 e^{ax} \, dx = ? \]
Trigonometric Integrals related to Fourier Series

Practice 5 (Trigonometric Integrals related to Fourier Series)

1. When \( m \neq n \), find \( \int_{-T_0/2}^{T_0/2} \cos \left( \frac{2\pi}{T_0} t \right) \cdot \cos \left( \frac{2\pi}{T_0} \frac{n}{T_0} \right) t \, dt = ? \)

2. When \( m = n \), find \( \int_{-T_0/2}^{T_0/2} \cos \left( \frac{2\pi}{T_0} t \right) \cdot \cos \left( \frac{2\pi}{T_0} \frac{m}{T_0} \right) t \, dt = ? \)

3. When \( m \neq n \), find \( \int_{-T_0/2}^{T_0/2} \sin \left( \frac{2\pi}{T_0} t \right) \cdot \sin \left( \frac{2\pi}{T_0} \frac{n}{T_0} \right) t \, dt = ? \)

4. When \( m = n \), find \( \int_{-T_0/2}^{T_0/2} \sin \left( \frac{2\pi}{T_0} t \right) \cdot \sin \left( \frac{2\pi}{T_0} \frac{m}{T_0} \right) t \, dt = ? \)

5. When \( m \neq n \) or \( m = n \), find \( \int_{-T_0/2}^{T_0/2} \cos \left( \frac{2\pi}{T_0} t \right) \cdot \sin \left( \frac{2\pi}{T_0} \frac{n}{T_0} \right) t \, dt = ? \)

Note: See if you know how to solve these. We will find out how to do it quickly in Sec. 11.1.
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Chapter 11. Fourier Analysis

- This chapter on Fourier analysis covers three broad areas:
  - Fourier series in Secs. 11.1–11.4
  - More general orthonormal series called Sturm–Liouville expansions in Secs. 11.5 and 11.6
  - Fourier integrals and transforms in Secs. 11.7–11.9

- The central starting point of Fourier analysis is Fourier series.
  - They are infinite series designed to represent general periodic functions in terms of simple ones, namely, cosines and sines.
  - This trigonometric system is orthogonal, allowing the computation of the coefficients of the Fourier series by use of the well-known Euler formulas.
  - Fourier series are very important to the engineer and physicist because they allow
    - the solution of ODEs in connection with forced oscillations (Sec. 11.3)
    - the approximation of periodic functions (Sec. 11.4)
    - applications of Fourier analysis to PDEs (Chap. 12)
Chapter 11. Fourier Analysis

- The underlying idea of the Fourier series can be extended in two important ways
  - **Replace the trigonometric system by other families of orthogonal functions:**
    - ✓ Sturm–Liouville expansions (Sec. 11.5)
    - ✓ Bessel functions (Sec. 11.6)
  - **Applying Fourier series to non-periodic phenomena** to obtain:
    - ✓ Fourier integrals (Sec. 11.7, 11.8)
    - ✓ Fourier transforms (Sec. 11.9)
  - Note: Both extensions have important applications to solving PDEs as will be shown in Chap. 12.

- In a digital age, the discrete Fourier transform plays an important role.
  - Signals, such as voice or music, are sampled and analyzed for frequencies.
  - An important algorithm, in this context, is the fast Fourier transform (Sec. 11.9)
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11.1 Fourier Series – Overview

• Purpose of Sec. 11.1:
  - To derive the Euler formulas for the coefficients of a Fourier series of a given function of period $2\pi$, using as the key property the orthogonality of the trigonometric system.

• Important Concepts of Sec. 11.1:
  - Periodic function
  - Trigonometric system, its orthogonality (Theorem 1)
  - Fourier series with Fourier coefficients and Euler formulas
  - Representation by a Fourier series (Theorem 2)

• Fourier Series
  - infinite series that represent periodic functions in terms of cosines and sines.
11.1 Fourier Series – Periodic Function

Definition 6 (Periodic Function)

A function $f(x)$ is called a periodic function if $f(x)$ is defined for all real $x$, except possibly at some points, and if there is some positive number $p$, called a period of $f(x)$, such that

$$f(x + p) = f(x) \quad \forall x$$  \hfill (1)

**Figure 6**: Periodic function of period $p$

- Example of periodic function: cosine, sine, tangent, and cotangent
  - $f(x) = \tan x$ is a periodic function that is not defined for all real $x$ but undefined for some points (more precisely, countably many points), i.e., $x = \pm \pi/2, \pm 3\pi/2$ ...

- Example of non-periodic function: $x$, $x^2$, $x^3$, $e^x$, $\cosh x$, and $\ln x$, ...
11.1 Fourier Series – Periodic Function

- If $f(x)$ has period $p$, it also has the period $2p$

\[ f(x + 2p) = f([x + p] + p) = f(x + p) = f(x) \]  \hspace{1cm} (2)

thus for any integer $n = 1, 2, 3..., f(x + np) = f(x) \forall x$

- The smallest positive period is often called the fundamental period.

- If $f(x)$ and $g(x)$ have period $p$, then $a \cdot f(x) + b \cdot g(x)$ with any constants $a$ and $b$ also has the period $p$. 
11.1 Fourier Series – Representation of Various functions of Period $2\pi$

- Represent various functions $f(x)$ of period $2\pi$ in terms of the simple functions:

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx, \ldots$$  \hspace{1cm} (3)

- All these functions have the period $2\pi$
  - constant 1: periodic with any period, its fundamental period not defined
- They form the so-called trigonometric system

![Cosine and sine functions](image)

Figure 7: Cosine and sine functions having the period $2\pi$ (the first few members of the trigonometric system, except for the constant 1)
11.1 Fourier Series – Fourier Series of \( f(x) \) with period \( 2\pi \)

- A trigonometric series:

\[
a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots + a_n \cos nx + b_n \sin nx + \ldots
\]

\[
= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]

- \( a_0, a_1, b_1, \ldots \) are constants, called the coefficients of the series.
- Each term has the period \( 2\pi \).

✓ Hence if the coefficients are such that the series converges, its sum will be a function of period \( 2\pi \).
11.1 Fourier Series – Fourier Series of $f(x)$ with period $2\pi$

**Definition 7 (Fourier Series of $f(x)$ with period $2\pi$)**

$f(x)$ is a given function of period $2\pi$ and is such that it can be represented by a trigonometric series, that is, it converges and has the sum $f(x)$. Then the Fourier series of $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$  \hspace{1cm} (4)

The coefficients are the Fourier coefficients of $f(x)$, given by the Euler formulas

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx$$  \hspace{1cm} (5)

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \quad n = 1, 2, \ldots$$  \hspace{1cm} (6)

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \quad n = 1, 2, \ldots$$  \hspace{1cm} (7)
Example 8 (ch 11.1, Example 1)

Find the Fourier coefficients of the periodic function $f(x)$ with period $2\pi$. The formula is

$$f(x) = \begin{cases} -k, & -\pi < x < 0 \\ k, & 0 < x < \pi \end{cases}$$

![Figure 8: Given function $f(x)$ (Periodic rectangular wave)](image)

- Functions of this kind occur as external forces acting on mechanical systems, electromotive forces in electric circuits, etc.
  - The value of $f(x)$ at a single point does not affect the integral; hence we can leave $f(x)$ undefined at $x = 0$ and $x = 2\pi$. 
11.1 Fourier Series – Example 1’s Answer

Answer:

\[ a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx = 0 \]

- The area under the curve of \( f(x) \) between \(-\pi \) and \( \pi \) (taken with a minus sign where \( f(x) \) is negative) is zero.
- DC value is zero.

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \left[ \int_{-\pi}^{0} (-k) \cos nx \, dx + \int_{0}^{\pi} k \cos nx \, dx \right] \]

\[ = \frac{1}{\pi} \left[ (-k) \left. \frac{\sin nx}{n} \right|_{-\pi}^{0} + k \left. \frac{\sin nx}{n} \right|_{0}^{\pi} \right] = 0 \]

- \( \sin nx = 0 \) at \(-\pi\), 0, and \( \pi \) for all \( n = 1, 2, \ldots \).
- All these cosine coefficients are zero. That is, the Fourier series of \( f(x) \) has no cosine terms, just sine terms,
11.1 Fourier Series – Example 1’s Answer

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \left( \int_{-\pi}^{0} (-k) \sin nx + \int_{0}^{\pi} k \sin nx \right) \]

\[ = \frac{1}{\pi} \left[ k \frac{\cos nx}{n} \bigg|_{-\pi}^{0} - k \frac{\cos nx}{n} \bigg|_{0}^{\pi} \right] \]

\[ = \frac{k}{n\pi} [\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0] = \frac{2k}{n\pi} (1 - \cos n\pi) \]

\[ \cos n\pi = \begin{cases} 
1, & \text{for } n = 0, 2, \ldots \text{ (even)} \\
-1, & \text{for } n = 1, 3, \ldots \text{ (odd)}
\end{cases} \quad \Rightarrow \quad 1 - \cos n\pi = \begin{cases} 
0, & \text{for } n = 0, 2, \ldots \text{ (even)} \\
2, & \text{for } n = 1, 3, \ldots \text{ (odd)}
\end{cases} \]

\[ \Rightarrow b_1 = \frac{4k}{\pi}, \quad b_3 = \frac{4k}{3\pi}, \quad b_5 = \frac{4k}{5\pi}, \ldots \]

\[ b_2 = b_4 = \ldots = 0 \]

\[ \Rightarrow f(x) = \frac{4k}{\pi} \left( \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x \ldots \right) \]
11.1 Fourier Series – Example 1’s Answer

- The figure right seems to indicate that the series is convergent and has the sum $f(x)$.
- Partial sum:
  \[ S_1 = \frac{4k}{\pi} \sin x, \]
  \[ S_2 = \frac{4k}{\pi} \left( \sin x + \frac{1}{3} \sin 3x \right) \]
  - At $x = 0$ and $x = \pi$, the points of discontinuity of $f(x)$, all partial sums have the value zero.
- Assuming that $f(x)$ is the sum of the series and setting $x = \pi/2$, we have
  \[ f\left(\frac{\pi}{2}\right) = k = \frac{4k}{\pi} \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right) \]
  - Thus, $\frac{\pi}{4} = \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right)$
  - A famous result obtained by Leibniz in 1673 from geometric considerations.
11.1 Fourier Series – Derivation of the Euler Formulas

The key to the Euler formulas is the orthogonality of the trigonometric system. Here we generalize the concept of inner product to functions.

Theorem 9 (Sec. 11.1 Theorem 1, Orthogonality of the Trigonometric System)

The trigonometric system \(1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots\) is orthogonal on the interval \(-\pi \leq x \leq \pi\) (hence also on \(0 \leq x \leq 2\pi\) or any other interval of length \(2\pi\) because of periodicity); that is, the integral of the product of any two functions in the trigonometric system over that interval is 0, so that for any integers \(n\) and \(m\),

(a) \[\int_{-\pi}^{\pi} \cos nx \cos mxdx = 0 \quad n \neq m\]

(b) \[\int_{-\pi}^{\pi} \sin nx \sin mxdx = 0 \quad n \neq m\]

(c) \[\int_{-\pi}^{\pi} \sin nx \cos mxdx = 0 \quad n \neq m \text{ or } n = m\]
This follows simply by transforming the integrands trigonometrically from products into sums. For (a) and (b), since \( m \neq n \) (integer!),

\[
\int_{-\pi}^{\pi} \cos nx \cos mxdx = \frac{1}{2} \int_{-\pi}^{\pi} \cos(n + m)xdx + \frac{1}{2} \int_{-\pi}^{\pi} \cos(n - m)xdx = 0
\]

\[
\int_{-\pi}^{\pi} \sin nx \sin mxdx = \frac{1}{2} \int_{-\pi}^{\pi} \cos(n - m)xdx - \frac{1}{2} \int_{-\pi}^{\pi} \cos(n + m)xdx = 0
\]

For (c), all integer \( m \) and \( n \),

\[
\int_{-\pi}^{\pi} \sin nx \cos mxdx = \frac{1}{2} \int_{-\pi}^{\pi} \sin(n + m)xdx + \frac{1}{2} \int_{-\pi}^{\pi} \sin(n - m)xdx = 0
\]
11.1 Fourier Series – Application of Theorem 1 to the Fourier Series

Proof

\[ f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \]

Integrate on both sides from \(-\pi\) to \(\pi\),

\[ \int_{-\pi}^{\pi} f(x) dx = \int_{-\pi}^{\pi} \left[ a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] dx \]

Assume that termwise integration is allowed,

\[ \int_{-\pi}^{\pi} f(x) dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left[ a_n \int_{-\pi}^{\pi} \cos nx dx + b_n \int_{-\pi}^{\pi} \sin nx dx \right] \]

\[ \Rightarrow a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \]
11.1 Fourier Series – Application of Theorem 1 to the Fourier Series

\[ f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \]

Similarly, multiplying on both sides by \( \cos mx \) with any fixed positive integer \( m \) and integrating from \( \pi \) to \( -\pi \),

\[ \int_{-\pi}^{\pi} f(x) \cos mx \, dx = a_0 \int_{-\pi}^{\pi} \cos mx \, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx \, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos mx \, dx \]

\[ \Rightarrow \int_{-\pi}^{\pi} f(x) \cos mx \, dx = a_m \pi \]

\[ \Rightarrow a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx \, dx \]
11.1 Fourier Series – Application of Theorem 1 to the Fourier Series

\[ f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \]

Multiplying on both sides by \( \sin mx \) with any fixed positive integer \( m \) and integrating from \( \pi \) to \( -\pi \),

\[ \int_{-\pi}^{\pi} f(x) \sin mx \, dx = a_0 \int_{-\pi}^{\pi} \sin mx \, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \sin mx \, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \sin mx \, dx \]

\[ \Rightarrow \int_{-\pi}^{\pi} f(x) \sin mx \, dx = b_m \pi \]

\[ \Rightarrow b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, dx \]
11.1 Fourier Series – Convergence and Sum of a Fourier Series

Definition 10 (Piecewise Continuous)

$f(x)$ is piecewise continuous on an interval $a \leq x \leq b$ where $f$ is defined. If this interval can be divided into finitely many subintervals in each of which $f$ is continuous and has finite limits as $x$ approaches either endpoint of such a subinterval from the interior. This then gives finite jumps as in the following figure as the only possible discontinuities.

Figure 10: Example of a piecewise continuous function $f(x)$. (The dots mark the function values at the jumps.)
11.1 Fourier Series – Convergence and Sum of a Fourier Series

Theorem 11 (Representation by a Fourier Series)

Let $f(x)$ be periodic with period $2\pi$ and piecewise continuous in the interval $-\pi \leq x \leq \pi$. Furthermore, let $f(x)$ have a left-hand derivative and a right-hand derivative at each point of that interval. Then

1. the Fourier series of $f(x)$ [with coefficients] converges. Its sum is $f(x)$, except at points $x_0$ where $f(x)$ is discontinuous.
2. the sum of the series is the average of the left- and right-hand limits of $f(x)$ at $x_0$.

Figure 11: Left-hand limit $f(1 + 0) = 1$ and right-hand limit $f(1 - 0) = \frac{1}{2}$ of the function $f(x) = \begin{cases} x^2, & \text{if } x < 1 \text{ (even)} \\ \frac{x}{2}, & \text{if } x \geq 1 \text{ (odd)} \end{cases}$
Recall the periodic rectangular wave example, where \( f(x) \) is piecewise continuous except \( x = 0, \pm \pi \ldots \)

\[
\begin{align*}
\text{Fourier series: } f(x) &= \frac{4k}{\pi} \left( \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x \ldots \right) \\
\text{At the discontinuous point } x &= 0, \text{ where the average of its left-hand limit (} -k \text{) and its right-hand limit (} k \text{) is 0, agree with the result of the sum of the series } f(0) = 0.
\end{align*}
\]
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11.2 Arbitrary Period. Even and Odd Functions. Half-Range Expansions—Overview

- **Purpose of Sec. 11.2:**
  1. Transition from period $2\pi$ to any period $2L$, for the function $f$, simply by a transformation of scale on the x-axis.
  2. **Simplifications:**
     1. Fourier cosine series: only cosine terms if $f$ is even
     2. Fourier sine series: only sine terms if $f$ is odd
  3. Half-range expansions: expansion of $f$ given for $0 \leq x \leq L$ in two Fourier series, one having only cosine terms and the other only sine terms

- **Important Concepts of Sec. 11.2:**
  - Fourier series with Fourier coefficients with Arbitrary Period
  - Simplifications of Fourier Series for Even/Odd Functions
11.2 Arbitrary Period – From Period $2\pi$ to Any Period $p = 2L$

- The period $2\pi$ in Sec. 11.1 has simple formulas.
- We discuss an arbitrary period $p = 2L$ in Sec. 11.2.
- **Change of Scale:** $f(v)$ period $2\pi \rightarrow f(x)$ period $2L$

$$\frac{v}{2\pi} = \frac{x}{2L} \Rightarrow v = \frac{\pi}{L}x \left( dv = \frac{\pi}{L}dx \right), f(v) = f\left(\frac{\pi}{L}x\right)$$

$$f(v) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nv + b_n \sin nv) \quad f\left(\frac{\pi}{L}x\right) = a_0 + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi}{L}x + b_n \sin \frac{n\pi}{L}x)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(v) dv \quad \Rightarrow \quad a_0 = \frac{1}{2L} \int_{-L}^{L} f\left(\frac{\pi}{L}x\right) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(v) \cos nv dv \quad n = 1, 2, \ldots$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f\left(\frac{\pi}{L}x\right) \cos \frac{n\pi}{L}x dx \quad n = 1, 2, \ldots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(v) \sin nv dv \quad n = 1, 2, \ldots$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f\left(\frac{\pi}{L}x\right) \sin \frac{n\pi}{L}x dx \quad n = 1, 2, \ldots$$
11.2 Arbitrary Period – From Period $2\pi$ to Any Period $p = 2L$

Let $F(x) \triangleq f\left(\frac{\pi}{L}x\right)$

\[ F(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right) \]

Replace $F(x)$ with $f(x)$

\[ f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right) \]

\[ a_0 = \frac{1}{2L} \int_{-L}^{L} F(x) \, dx \]

\[ a_n = \frac{1}{L} \int_{-L}^{L} F(x) \cos \frac{n\pi}{L} x \, dx \quad n = 1, 2, \ldots \]

\[ b_n = \frac{1}{L} \int_{-L}^{L} F(x) \sin \frac{n\pi}{L} x \, dx \quad n = 1, 2, \ldots \]

\[ \Rightarrow \quad a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx \]

\[ a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x \, dx \quad n = 1, 2, \ldots \]

\[ b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x \, dx \quad n = 1, 2, \ldots \]
11.2 Arbitrary Period – Fourier Series of \( f(x) \) with period \( 2L \)

**Definition 13 (Fourier Series of \( f(x) \) with period \( 2L \))**

\( f(x) \) is a given function of period \( 2L \) and is such that it can be represented by a trigonometric series, that is, it converges and has the sum \( f(x) \). Then the Fourier series of \( f(x) \) is

\[
f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)
\]

The coefficients are the Fourier coefficients of \( f(x) \), given by the **Euler formulas**

\[
a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx
\]

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x \, dx \quad n = 1, 2, \ldots
\]

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x \, dx \quad n = 1, 2, \ldots
\]
11.2 Arbitrary Period – Example 1

Example 14 (Sec. 11.2 Example 1, Periodic Rectangular Wave)

Find the Fourier series of the function

\[ f(x) = \begin{cases} 
0, & -2 < x < -1 \\
k, & -1 < x < 1 \\
0, & 1 < x < 2 
\end{cases} \]

where \( p = 4 \).
11.2 Arbitrary Period – Example 1’s Answer

\[ p = 4 \Rightarrow L = 2 \]

\[ a_0 = \frac{k}{2} \]

\[ a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x \, dx = \frac{1}{2} \int_{-2}^{2} f(x) \cos \frac{n\pi}{2} x \, dx \]

\[ = \frac{1}{2} \frac{1}{n\pi} k \sin \frac{n\pi}{2} \left|^{1}_{-1} = \frac{2k}{n\pi} \sin \frac{n\pi}{2} = \begin{cases} 0, & n \text{ is even} \\ \frac{2k}{n\pi}, & n = 1, 5, 9, \ldots \\ \frac{2k}{n\pi}, & n = 3, 7, 11, \ldots \end{cases} \]

\[ b_n = \frac{1}{2} \int_{-1}^{1} k \sin \frac{n\pi}{2} x \, dx = 0 \]

\[ \Rightarrow f(x) = \frac{k}{2} + \frac{2k}{\pi} \left( \cos \frac{\pi}{2} - \frac{1}{3} \cos \frac{3\pi}{2} + \frac{1}{5} \cos \frac{5\pi}{2} \ldots \right) \]
Find the Fourier series of the function \( f(x) = \begin{cases} -k, & -2 < x < -2 \\ k, & 0 < x < 2 \end{cases} \), where \( p = 4 \).
11.2 Arbitrary Period – Example 2’s Answer

Recall Example 1 of Sec. 11.1,

\[ g(v) = \begin{cases} 
-k, & -\pi < v < 0 \\
 k, & 0 < v < \pi 
\end{cases} = \frac{4k}{\pi} \left( \sin v + \frac{1}{3} \sin 3v + \frac{1}{5} \sin 5v \ldots \right) \]

now \( \frac{v}{2\pi} = \frac{x}{2L} \Rightarrow v = \frac{\pi}{L}x = \frac{\pi}{2}x \)

\[ \Rightarrow f(x) = \frac{4k}{\pi} \left( \sin \frac{\pi}{2}x + \frac{1}{3} \sin \frac{3\pi}{2}x + \frac{1}{5} \sin \frac{5\pi}{2}x \ldots \right) \]
Example 16 (ch 11.2, Example 3, Half-Wave Rectifier)

A sinusoidal voltage $E \cdot \sin \omega t$, where $t$ is time, is passed through a half-wave rectifier that clips the negative portion of the wave. Find the Fourier series of the resulting periodic function.

$$u(t) = \begin{cases} 
0, & -L < t < 0 \\
E \cdot \sin \omega t, & 0 < t < L 
\end{cases}$$

where $p = 2L = \frac{2\pi}{\omega}$. 
11.2 Arbitrary Period – Example 3’s Answer

Answers.

\[ a_0 = \frac{\omega}{2\pi} \int_0^{\pi/\omega} E \sin \omega t \, dt = \frac{\omega}{2\pi} E \left[ \frac{1}{\omega} (-\cos \omega t) \right]_0^{\pi/\omega} = \frac{E}{\pi} \]

\[ a_n = \frac{\omega}{\pi} \int_0^{\pi/\omega} E \sin \omega t \cos n\omega t \, dt = \frac{\omega E}{2\pi} \int_0^{\pi/\omega} \left[ \sin((1+n)\omega t + \sin((1-n)\omega t) \right] \, dt \]

\[ \Rightarrow a_1 = \frac{\omega E}{2\pi} \int_0^{\pi/\omega} \left[ \sin((1+1)\omega t + \sin((1-1)\omega t) \right] \, dt = \frac{\omega E}{2\pi} \int_0^{\pi/\omega} \sin 2\omega t \, dt = \frac{E}{4\pi} (-\cos 2\omega t) \left|_0^{\pi/\omega} \right. = 0 \]

\[ \Rightarrow a_n = \frac{\omega E}{2\pi} \left[ -\frac{\cos((1+n)\omega t)}{(1+n)\omega} - \frac{\cos((1-n)\omega t)}{(1-n)\omega} \right] = \frac{\omega E}{2\pi} \left[ -\frac{\cos((1+n)\pi)}{(1+n)\omega} + 1 - \frac{\cos((1-n)\pi)}{(1-n)\omega} + 1 \right] \]

\[ = \begin{cases} 
0, & \text{n is odd} \\
\frac{E}{2\pi} \left[ \frac{2}{1+n} + \frac{2}{1-n} \right], & \text{n is even}
\end{cases} \]
11.2 Arbitrary Period – Example 3’s Answer

Answers.

\[ b_n = \frac{\omega}{\pi} \int_{0}^{\pi} E \sin \omega t \sin n\omega t \, dt = \frac{\omega E}{2\pi} \int_{0}^{\pi} \left[ \cos(1 - n)\omega t - \cos(1 + n)\omega t \right] \, dt \]

\[ \Rightarrow b_n = \frac{\omega E}{2\pi} \left[ \frac{\sin(1 - n)\pi}{(1 - n)\omega} - \frac{\sin(1 + n)\pi}{(1 + n)\omega} \right] \]

\[ = \begin{cases} \frac{\omega E}{2\pi} \frac{\pi}{\omega} - \frac{E \sin 2\pi}{2\pi} 2 = \frac{E}{2}, & n = 1 \\ 0, & n = 2, 3, \ldots \end{cases} \]
11.2 Simplifications: Even and Odd Functions

For even \( g \), \( g(x) = g(-x) \)
\[
\Rightarrow \int_{-L}^{L} g(x) \, dx = 2 \int_{0}^{L} g(x) \, dx
\]

For odd \( g \), \( g(x) = -g(-x) \)
\[
\Rightarrow \int_{-L}^{L} g(x) \, dx = 0
\]
\[\Rightarrow \text{No DC Value}\]
11.2 Simplifications: Even and Odd Functions

Definition 17 (Fourier Cosine Series)

If $f(x)$ is an even function, that is, $f(x) = f(-x)$, its Fourier series reduces to a Fourier cosine series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L}x$$

where

$$a_0 = \frac{1}{L} \int_{0}^{L} f(x) dx$$

$$a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi}{L}x dx \quad n = 1, 2, ...$$

- $f(x) \cdot \cos \frac{n\pi}{L} \rightarrow \text{even} \Rightarrow \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L}x dx = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi}{L}x dx$

- $f(x) \cdot \sin \frac{n\pi}{L} \rightarrow \text{odd} \Rightarrow \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L}x dx = 0$
11.2 Simplifications: Even and Odd Functions

**Definition 18 (Fourier Sine Series)**

If $f(x)$ is an even function, that is, $f(x) = f(-x)$, its Fourier series reduces to a Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

where

$$b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi}{L} x \, dx \quad n = 1, 2, \ldots$$

- $f(x) \cdot \cos \frac{n\pi}{L} \rightarrow$ odd $\Rightarrow \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x \, dx = 0$
- $f(x) \cdot \sin \frac{n\pi}{L} \rightarrow$ even $\Rightarrow \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x \, dx = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi}{L} x \, dx$
11.2 Simplifications: Example 4

Example 19 (Fourier Cosine and Sine Series)

In Example 1, an even function, $b_n = 0$

$$f(x) = \frac{k}{2} + \frac{2k}{\pi} \left( \cos \frac{\pi}{2} - \frac{1}{3} \cos \frac{3\pi}{2} + \frac{1}{5} \cos \frac{5\pi}{2} \cdots \right)$$

In Example 2, an odd function, $a_0 = a_n = 0$

$$f(x) = \frac{4k}{\pi} \left( \sin \frac{\pi}{2} x + \frac{1}{3} \sin \frac{3\pi}{2} x + \frac{1}{5} \sin \frac{5\pi}{2} x \cdots \right)$$
11.2 Simplifications: Sum and Scalar Multiple

**Definition 20 (Sum and Scalar Multiple)**

1. **Sum**: the Fourier coefficients of a sum $f_1 + f_2$ are the sums of the corresponding Fourier coefficients of $f_1$ and $f_2$.
2. **Scalar Multiple**: the Fourier coefficients of $cf$ are $c$ times the corresponding Fourier coefficients of $f$.

**Note:**
- We often refers the sum and scalar multiple as the linear property.
- If we know the Fourier series coefficients $(a_0, a_n, b_n)$ of $f_1$ and that $(\bar{a}_0, \bar{a}_n, \bar{b}_n)$ of $f_2$, then we know the Fourier series coefficients of any linear combination $cf_1 + df_2$ as $ca_0 + d\bar{a}_0, ca_n + d\bar{a}_n, cb_n + \bar{b}_n$. 

![Image of the page](image_url)
11.2 Simplifications: Example 5

Example 21 (Sec. 11.2, Example 5, Sawtooth Wave)

Find the Fourier series of the function

\[ f(x) = x + \pi \text{ if } -\pi < x < \pi \text{ and } f(x + 2\pi) = f(x) \]
11.2 Simplifications: Example 5’s Answer

\[ f(x) = x + \pi \Rightarrow \begin{cases} 
  f_1(x) = x \text{ (odd)} \\
  f_2(x) = \pi \text{ (even)} 
\end{cases} \]

\( f_1: \) odd function
\[ \Rightarrow a_0 = a_n = 0 \]
\[ \Rightarrow b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \, dx = \frac{2}{\pi} \int_{0}^{\pi} x \sin nx \, dx \]
\[ = \frac{2}{\pi} \left[ -x \cos nx \frac{1}{n} \right]_{0}^{\pi} + \frac{1}{n} \int_{0}^{\pi} \cos nx \, dx \]
\[ = \frac{2}{\pi} (-\pi) \frac{\cos n\pi}{n} = -\frac{2}{n} \cos n\pi \]

\( f_2: \) constant
\[ \Rightarrow a_0 = \pi, a_n = b_n = 0 \]

\( f_1 + f_2: \)
\[ \Rightarrow f(x) = \pi + 2 \left( \sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \ldots \right) \]
11.2 Half-Range Expansions

Figure 13: The given function $f(x)$

Figure 14: $f(x)$ continued as an even periodic function of period $2L$.

Figure 15: $f(x)$ continued as an odd periodic function of period $2L$. 
11.2 Half-Range Expansions: Example 6

Example 22 (Sec. 11.2, Example 6, “Triangle” and Its Half-Range Expansions)

Find the two half-range expansions of the function \( f(x) = \begin{cases} 
\frac{2k}{L} x, & 0 < x < \frac{L}{2} \text{ (odd)} \\
\frac{2k}{L} (L - x), & \frac{L}{2} < x < L \text{ (even)} 
\end{cases} \)

Figure 16: \( f(x) \)
11.2 Half-Range Expansions: Example 6’s Answer

(a) even periodic extension: note that the answer is equivalent to the Fourier cosine series of $f(x)$

\[
a_0 = \frac{1}{L} \left[ \frac{2k}{L} \int_0^{L/2} x \, dx + \frac{2k}{L} \int_{L/2}^L (L - x) \, dx \right] = \frac{k}{2}
\]

Figure 17: Even extension of $f(x)$
11.2 Half-Range Expansions: Example 6’s Answer

\[ a_n = \frac{2}{L} \left[ \frac{2k}{L} \int_0^{L/2} x \cos \frac{n\pi x}{L} \, dx + \frac{2k}{L} \int_{L/2}^{L} (L - x) \cos \frac{n\pi x}{L} \, dx \right] \]

where

\[ \int_0^{L/2} x \cos \frac{n\pi x}{L} \, dx = \frac{Lx}{n\pi} \sin \frac{n\pi x}{L} \bigg|_0^{L/2} - \frac{L}{n\pi} \int_0^{L/2} \sin \frac{n\pi x}{L} \, dx = \frac{L^2}{2n\pi} \sin \frac{n\pi}{2} + \frac{L^2}{n^2\pi^2} (\cos \frac{n\pi}{2} - 1) \]

and

\[ \int_{L/2}^{L} (L - x) \cos \frac{n\pi x}{L} \, dx = \frac{L}{n\pi} (L - x) \sin \frac{n\pi x}{L} \bigg|_{L/2}^{L} + \frac{L}{n\pi} \int_{L/2}^{L} \sin \frac{n\pi x}{L} \, dx \]

\[ = -\frac{L}{n\pi} \frac{L}{2} \sin \frac{n\pi}{2} - \frac{L^2}{n^2\pi^2} \left( \cos n\pi - \cos \frac{n\pi}{2} \right) \]

\[ \Rightarrow a_n = \frac{4k}{n^2\pi^2} \left( 2 \cos \frac{n\pi}{2} - \cos n\pi - 1 \right) \Rightarrow a_2 = \frac{-16k}{4\pi^2}, a_6 = \frac{-16k}{6\pi^2} \]

\[ \Rightarrow f(x) = \frac{k}{2} - \frac{16k}{\pi^2} \left( \frac{1}{2^2} \cos \frac{2\pi}{L} x + \frac{1}{6^2} \cos \frac{6\pi}{L} x + \ldots \right) \]
(b) odd periodic extension: note that the answer is equivalent to the Fourier sine series of \( f(x) \).

\[
b_n = \frac{8k}{n^2\pi^2} \sin \frac{n\pi}{2}
\]

\[
f(x) = \frac{8k}{\pi^2} \left( \frac{1}{1^2} \sin \frac{\pi}{L} x - \frac{1}{3^2} \sin \frac{3\pi}{L} x + \ldots \right)
\]
Write down math expression of $f(x)$.

1. Find the Fourier series of function $f(x)$ by regarding the period as $2L$. [Note: the answer is equivalent to the Fourier cosine series of $f(x)$.]

2. Find the Fourier series of function $f(x)$ by regarding the period as $L$.

3. Compare (2) and (3) and conclude your findings.
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Forced oscillations of a body of mass $m$ on a spring of modulus $k$ are governed by the ODE

$$my'' + cy' + ky = r(t)$$

- $y(t)$: the displacement from rest
- $c$: the damping constant

✓ If $r(t)$ is a sine or cosine function and if there is damping ($c > 0$), then the steady-state solution is a harmonic oscillation with frequency equal to that of $r(t)$.

✓ What if $r(t)$ is any other periodic function?

- $k$: the spring constant (spring modulus)
- $r(t)$: the external force depending on time $t$.

The steady-state solution will be a superposition of harmonic oscillations with frequencies equal to that of $r(t)$ and integer multiples of these frequencies.

If one of these frequencies is close to the resonant frequency of the vibrating system, then the corresponding oscillation may be the dominant part of the response of the system to the external force.
11.3 Forced Oscillations: Example 1

Example 24 (Sec. 11.3 Example 1, Forced Oscillations under a Nonsinusoidal Periodic Driving Force)

Let \( m = 1 \, (g) \), \( c = 0.05 \, (g/sec) \), and \( k = 25 \, (g/sec^2) \), so that

\[
y'' + 0.05y' + 25y = r(t),
\]

where \( r(t) \) is measured in \((g \cdot cm/sec^2)\)

\[
r(t) = \begin{cases} 
  t + \frac{\pi}{2}, & -\pi < t < 0 \\
  -t + \frac{\pi}{2}, & 0 < t < \pi
\end{cases}
\]

and \( r(t) = r(t + 2\pi) \).

Find the steady state solution \( y(t) \).
Method 1:
Represent $r(t)$ by a Fourier series:

$$r(t) = \frac{\pi}{4} \left( \cos t + \frac{1}{3^2} \cos 3t + \frac{1}{5^2} \cos 5t + \ldots \right)$$

Then the ODE

$$y'' + 0.05y' + 25y = \sum_{n=1,3,\ldots} \frac{\pi}{4n^2} \cos nt$$

the steady-state solution $y_n(t)$ is of the form

$$y_n = A_n \cos nt + B_n \sin nt$$

$$\Rightarrow A_n = \frac{4(25 - n^2)}{n^2 \pi [ (25 - n^2)^2 + (0.05n)^2 ]}, \quad B_n = \frac{0.2}{n \pi [ (25 - n^2)^2 + (0.05n)^2 ]}$$

Since the ODE is linear, the steady-state solution is $y = y_1 + y_3 + \ldots$. 
11.3 Forced Oscillations: Example 1’s Answer

How to plot the amplitude spectrum of the steady-state solution?

\[ y_n = A_n \cos nt + B_n \sin nt = C_n \cos(nt - \theta_n) \]

where

\[ C_n = \sqrt{A_n^2 + B_n^2}, \quad \theta_n = \tan^{-1} \frac{B_n}{A_n} \]

\[ \Rightarrow C_n = \sqrt{\frac{16[(25 - n^2)^2 + (0.05n)^2]}{n^4\pi^2[(25 - n^2)^2 + (0.05n)^2]^2}} = \frac{4}{n^2\pi \sqrt{(25 - n^2)^2 + (0.05n)^2}} \]

\[ \Rightarrow C_1 = 0.0531, \quad C_3 = 0.0088, \quad C_5 = 0.2037, \quad C_7 = 0.0011, \quad C_9 = 0.0003... \]
11.3 Forced Oscillations: Example 1’s Answer

method 2. complex phasor:

\[ y''_n + 0.05y'_n + 25y_n = \frac{4}{n^2\pi} \cos nt \]

let \( y_n = \Re\{Y_n e^{int}\}, \cos nt = \Re\{e^{int}\} \)

\[ \Rightarrow (jn)^2Y_ne^{int} + 0.05(jn)Y_ne^{int} + 25Y_ne^{int} = \frac{4}{n^2\pi}e^{int} \]

\[ \Rightarrow Y_n = \frac{4}{n^2\pi(25 - n^2) + 0.05jn} = \frac{4}{n^2\pi} \frac{(25 - n^2) - 0.05jn}{[(25 - n^2)^2 + (0.05n)^2]} \]

\[ \Rightarrow y(t) = \sum_n y_n(t) = \sum_n \Re\{Y_ne^{int}\} = \sum_{n=1,3,...} C_n \cos(nt - \theta_n) \]

where \( C_n = \sqrt{\frac{4^2}{(n^2\pi)^2} \frac{(25 - n^2)^2 + (0.05n)^2}{[(25 - n^2)^2 + (0.05n)^2]^2}} = \frac{4}{n^2\pi} \frac{1}{\sqrt{(25 - n^2)^2 + (0.05n)^2}} \)

\[ \theta_n = \tan^{-1} \frac{0.05n}{25 - n^2} \]
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11.4 Approximation by Trigonometric Polynomials

- Fourier series play a prominent role not only in differential equations but also in approximation theory
  - an area that is concerned with approximating functions by other functions—simpler functions

- Let \( f(x) \) be a function on the interval \(-\pi \leq x \leq \pi\) that can be represented on this interval by a Fourier series. The \( N\)th partial sum of the Fourier series

\[
f(x) \approx a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)
\]

is an approximation of the given \( f(x) \).

- Is it the “best” approximation of \( f \) by a trigonometric polynomial of the same degree \( N \)?

\[
F(x) = A_0 + \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx)
\]

- “best” means that the “error” of the approximation is as small as possible.
11.4 Approximation by Trigonometric Polynomials

- Define “error” of such an approximation:
  - In connection with Fourier series, it is better to choose a definition of error that measures the goodness of agreement between $f$ and $F$ on the whole interval $-\pi \leq x \leq \pi$.

1. Define the error as $E = \max |f(x) - F(x)|$

   $$\arg \min \max_F |f(x) - F(x)|$$

   subject to $-\pi \leq x \leq \pi$

   - the sum $f$ of a Fourier series may have jumps: $F$ is a good overall approximation of $f$, but the maximum of $|f(x) - F(x)|$ (more precisely, the supremum) is large.

2. Define the square error of $F$ relative to function $f$ on $-\pi \leq x \leq \pi$ as

   $E = \int_{-\pi}^{\pi} [f(x) - F(x)]^2 dx$

   $$\arg \min_F \int_{-\pi}^{\pi} [f(x) - F(x)]^2 dx$$

   subject to $-\pi \leq x \leq \pi$
11.4 Approximation by Trigonometric Polynomials

Theorem 25 (Sec. 11.4, Theorem 1, Minimum Square Error)

The square error of \( F \) in \( F(x) = A_0 + \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx) \) (with fixed \( N \)) relative to \( f \) on the interval \(-\pi \leq x \leq \pi\) is minimum if and only if the coefficients of \( F(A_0, A_n, B_n) \) are the Fourier coefficients of \( f \).

Example 26 (Sec. 11.4, Example 1, Minimum Square Error for the Sawtooth Wave)

Given \( f(x) = x + \pi \) \((-\pi < x < \pi\)), the minimum mean square error approximation \( F(x) \) with fixed \( N \) is \( F(x) = \pi + 2(\sin x - \frac{1}{2}\sin 2x + \ldots + \frac{(-1)^{N+1}}{N} \sin Nx) \).

Figure 23: \( F \) with \( N = 20 \)
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The idea of the Fourier series was to represent general periodic functions in terms of cosines and sines.

- A trigonometric system
- Orthogonality which allows the coefficient computation of the Fourier series by the Euler formulas

Can this approach be generalized? That is, can we replace the trigonometric system of Sec. 11.1 by other orthogonal systems (sets of other orthogonal functions)?

- The answer is “yes”
- It will lead to generalized Fourier series, including the Fourier–Legendre series and the Fourier–Bessel series in Sec. 11.6
- To prepare for this generalization, we first have to introduce the concept of a Sturm–Liouville Problem
11.5 Sturm–Liouville Problems

Problem 27 (Sturm–Liouville (SL) problem)

Consider a second-order ODE of the form

$$[p(x)y']' + [q(x) + \lambda r(x)]y = 0$$  \hspace{1cm} (9)

on some interval $a \leq x \leq b$, satisfying conditions of the form

$$k_1y + k_2y' = 0 \text{ at } x = a \quad (a)$$

$$l_1y + l_2y' = 0 \text{ at } x = b \quad (b)$$  \hspace{1cm} (10)

- Equation (9) is known as a Sturm–Liouville equation
- Sturm–Liouville problem is an example of a boundary value problem
  - A boundary value problem consists of an ODE and given boundary conditions referring to the two boundary points (endpoints) $x = a$ and $x = b$ of a given interval $a \leq x \leq b$
- $\lambda$ is a parameter; $k_1, k_2, l_1, l_2$ are given real constants
  - $y = 0$ is a “trivial solution” of the problem for any $\lambda$, which is of no interest
  - We want to find eigenfunctions $y(x)$ as solutions of (9) satisfying (10) without being identically zero
  - $\lambda$ for which an eigenfunction exists is an eigenvalue of the SL problem
  - $\Rightarrow$ Find the eigenfunctions ($y \neq 0$) and their corresponding eigenvalues
Example 28 (Sec. 11.5, Example 1, Trigonometric Functions as Eigenfunctions. Vibrating String)

Find the eigenvalues and eigenfunctions of the Sturm–Liouville problem

\[ y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) = 0 \]  

(11)

[Note: the “space function” of the deflection \( u(x, t) \) of the string vibration discussed in Sec. 12.3]

This is a SL problem since \( p = 1, q = 0, r = 1 \) in (9), and \( a = 0, b = \pi, k_1 = l_1 = 1, k_2 = l_2 = 0 \) in (10)
11.5 Sturm–Liouville Problems: Example 1’s Answer

(a) For negative $\lambda = -v^2$,

\[ s^2 + v^2 = 0 \Rightarrow s = \pm v \]

Therefore, the general solution,

\[ y(x) = c_1 e^{vx} + c_2 e^{-vx} \]

\[ y(0) = c_1 + c_2 = 0 \]
\[ y(\pi) = c_1 e^{v\pi} + c_2 e^{-v\pi} = 0 \]

\[ c_1 = c_2 = 0 \]

\[ \Rightarrow y = 0, \text{ which is not an eigenfunction} \]

(b) For $\lambda = 0 \Rightarrow y = 0$, which is not an eigenfunction
11.5 Sturm–Liouville Problems: Example 1’s Answer

(c) For positive $\lambda = +\nu^2$,
\[ \Rightarrow s^2 - \nu^2 = 0 \Rightarrow s = \pm iv \]

Therefore, the general solution,

\[ y(x) = A \cos \nu x + B \sin \nu x \]

\[ \Rightarrow \begin{cases} 
    y(0) = A + B \cdot 0 = 0 \\
    y(\pi) = A \cos \nu \pi + B \sin \nu \pi = 0 
\end{cases} \]

\[ A = 0 \Rightarrow B \sin \nu \pi = 0 \]

Note that $y$ is an eigenfunction if $B \neq 0$. Thus, $\nu = 0, \pm 1, \pm 2, ...$

1. for $\nu = 0 \Rightarrow B \sin 0 = 0$

2. for $\nu = \pm 1, \pm 2, \pm 3 ...$ (eigenvalues $\lambda = 1, 4, 9, ...$), we take $B = 1$
   \[ \Rightarrow \text{eigenfunctions } y = \sin \nu x (\sin \sqrt{\lambda} x) \]
The solution to this problem is precisely the trigonometric system of the Fourier series considered earlier.

The most remarkable and important property of eigenfunctions of Sturm–Liouville problems is their orthogonality.
11.5 Orthogonal Functions

**Definition 29 (Orthogonal functions)**

Functions $y_1(x), y_2(x), \ldots$ defined on some interval $a \leq x \leq b$ are called orthogonal on this interval with respect to the weight function $r(x) > 0$ if for all $m$ and all $n$ different from $m$,

$$ (y_m, y_n) = \int_a^b r(x)y_m(x)y_n(x)dx = 0 \quad (m \neq n) $$

**Definition 30 (Norm)**

The norm $\|y_m\|$ of $y_m$ is defined by

$$ \|y_m\| = \sqrt{(y_m, y_m)} = \sqrt{\int_a^b r(x)y_m^2(x)dx} $$
11.5 Orthogonal Functions

Definition 31 (Orthonormal functions)

The functions $y_1(x), y_2(x), \ldots$ are called orthonormal on $a \leq x \leq b$ if they are orthogonal on this interval and all have norm 1

$$ (y_m, y_n) = \int_a^b r(x)y_m(x)y_n(x)\,dx = \delta_{mn} = \begin{cases} 0, & \text{if } m \neq n \\ 1, & \text{if } m = n \end{cases} $$

- $\delta_{mn}$ is Kronecker symbol
- We more briefly call the functions orthogonal/orthonormal instead of orthogonal/orthonormal with respect to $r(x) = 1$
Show that the functions $y_m(x) = \sin mx, m = 1, 2, \ldots$ form an orthogonal set on the interval $-\pi \leq x \leq \pi$ and find their corresponding orthonormal set.

- For $m \neq n$,
  \[ (y_m, y_n) = \int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos(m-n)x \, dx - \frac{1}{2} \int_{-\pi}^{\pi} \cos(m+n)x \, dx = 0 \]

- For $m = n$, the norm
  \[ \|y_m\|^2 = (y_m, y_m) = \int_{-\pi}^{\pi} \sin^2 mx \, dx = \pi \]

Hence the corresponding orthonormal set, obtained by division by the norm, is
\[ \frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \frac{\sin 3x}{\sqrt{\pi}}, \ldots \]
11.5 Sturm–Liouville Problems. Orthogonal Functions

Theorem 33 (Sec. 11.5, Theorem 1, Orthogonality of Eigenfunctions of Sturm–Liouville Problems)

Suppose that the functions $p$, $q$, $r$, and $p'$ in the Sturm–Liouville equation (9) are real-valued and continuous and $r(x) > 0$ on the interval $a \leq x \leq b$. Let $y_m(x)$ and $y_n(x)$ be eigenfunctions of the Sturm–Liouville problem (9)(10) that correspond to different eigenvalues $\lambda_m$ and $\lambda_n$, respectively. Then $y_m, y_n$ are orthogonal on that interval with respect to the weight function $r$, that is,

$$ (y_m, y_n) = \int_a^b r(x)y_m(x)y_n(x)dx = 0 \quad (m \neq n) $$

If $p(a) = 0$, then (10)(a) can be dropped from the problem. If $p(b) = 0$, then (10)(b) can be dropped. [It is then required that $y$ and $y'$ remain bounded at such a point, and the problem is called singular, as opposed to a regular problem in which (10) is used.]

If $p(a) = p(b)$, then (10) can be replaced by the “periodic boundary conditions”

$$ y(a) = y(b), \quad y'(a) = y'(b) $$
11.5 Sturm–Liouville Problems. Orthogonal Functions

- Theorem 33 shows that for any Sturm–Liouville problem, the eigenfunctions associated with these problems are orthogonal.
  - This means, in practice, if we can formulate a problem as a Sturm–Liouville problem, then by this theorem we are guaranteed orthogonality.

Example 34 (Sec. 11.5, Example 3, Application of Theorem 1. Vibrating String)
The ODE in Example 28 is a Sturm–Liouville equation with \( p = 1, q = 0, \) and \( r = 1 \). From Theorem 33 it follows that the eigenfunctions \( y_m = \sin mx (m = 1, 2, \ldots) \) are orthogonal on the interval \( 0 \leq x \leq \pi \).
11.5 Exercise

Practice 35 (SL problem)

Find the eigenvalues and eigenfunction of the Sturm-Liouville problem.

1. $y'' + \lambda y = 0, y(0) = 0, y(\pi) = 0$
2. $y'' + \lambda y = 0, y'(0) = 0, y(L) = 0$
3. $y'' + \lambda y = 0, y(0) = 0, y'(L) = 0$
4. What is the most remarkable and important property of eigenfunctions of Sturm-Liouville problems?
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Let $y_0, y_1, y_2$ be orthogonal with respect to a weight function $r(x)$ on an interval $a \leq x \leq b$, and let $f(x)$ be a function that can be represented by a convergent series

$$f(x) = \sum_{m=0}^{\infty} a_m y_m(x) = a_0y_0(x) + a_1y_1(x) + \ldots$$

This is called an orthogonal series, orthogonal expansion, or generalized Fourier series.

If the $y_m$ are the eigenfunctions of a Sturm–Liouville problem, we call it an eigenfunction expansion.
11.6 Orthogonal Series. Generalized Fourier Series

- Given \( f(x) \), we have to determine the coefficients in (12), called the Fourier constants of \( f(x) \) with respect to \( y_0, y_1 \).
  - We multiply both sides of (12) by \( r(x)y_n(x) \)

\[
(f, y_n) = \int_a^b \! r f y_n \, dx = \int_a^b \! r \left( \sum_{m=0}^{\infty} a_m y_m \right) y_n = \sum_{m=0}^{\infty} a_m \int_a^b \! y_m y_n \, dx = \sum_{m=0}^{\infty} a_m (y_m, y_n)
\]

- Because of the orthogonality all the integrals on the right are zero, except when \( m = n \)

\[
(f, y_n) = a_n (y_n, y_n) = a_n \| y_n \|^2
\]

- We get the desired formula for the Fourier constants

\[
a_m = \frac{(f, y_m)}{\| y_m \|^2} = \frac{1}{\| y_m \|^2} \int_a^b \! r(x)f(x)y_m(x) \, dx
\]

This formula generalizes the Euler formulas in Sec. 11.1 by orthogonality.
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11.7 Fourier Integral

- Fourier Series: periodic, interests on a finite interval
- Fourier Integral: non-periodic, interests on the whole x-axis

**Fourier integrals**: many problems involve functions that are nonperiodic and are of interest on the whole x-axis, we ask what can be done to extend the method of Fourier series to such functions

- In Example 36 we start from a **special** function $f_L$ of period $2L$ and see what happens to its Fourier series if we let $L \to \infty$
- Then we do the same for an **arbitrary** function $f_L$ of period $2L$
- This will motivate and suggest the main result of this section, which is an integral representation given in Theorem 37
Consider the periodic rectangular wave $f_L(x)$ of period $2L > 2$ given by

$$f(x) = \begin{cases} 
0, & -L < x < -1 \\
1, & -1 < x < 1 \\
0, & 1 < x < L 
\end{cases}$$

Explore what happens to the Fourier coefficients of $f_L$ as $L$ increases, for $2L = 4, 8, 16$ as well as the nonperiodic function $f(x)$, which we obtain from $f_L$ if we let $L \to \infty$.

$f_L$ is an even function $\Rightarrow b_n = 0$

$$a_0 = \frac{1}{2L} \int_{-1}^{1} dx = \frac{1}{L}$$

$$a_n = \frac{1}{L} \int_{-1}^{1} \cos \frac{n\pi}{L} x dx = \frac{2}{L} \int_{0}^{1} \cos \frac{n\pi}{L} x dx = \frac{2}{L} \frac{\sin \frac{n\pi}{L}}{\frac{n\pi}{L}}$$
11.7 Fourier Integral: Example 1’s Answer

- **Amplitude spectrum** of $f_L$: Fourier coefficients of $f_L$ in this example, where $|a_n|$ is the maximum amplitude of the wave $a_n \cos(n\pi x/L)$
- When $L \to \infty$ $\Rightarrow$ these amplitudes become more and more **dense** on the positive $w_n$-axis, where $w_n = \frac{n\pi}{L}$

Figure 24: Waveforms and amplitude spectra in Example 36
Now consider any periodic function $f_L(x)$ of period $2L$ that can be represented by a Fourier series and find out what happens if we let $L \to \infty$

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos w_n x + b_n \sin w_n x), \quad w_n = \frac{n\pi}{L}$$

- An integral (instead of a series) involving $\cos wx$ and $\sin wx$ with $w$ no longer restricted to integer multiples $w = w_n = n\pi/L$ of $\pi/L$ but taking all values

If we insert $a_n$ and $b_n$ from the Euler formulas of Sec. 11.2, and denote the variable of integration by $v$

$$f_L(x) = \frac{1}{2L} \int_{-L}^{L} f_L(v) dv + \frac{1}{L} \sum_{n=1}^{\infty} \left[ \cos w_n x \int_{-L}^{L} f_L(v) \cos w_n v dv + \sin w_n x \int_{-L}^{L} f_L(v) \sin w_n v dv \right]$$

We now set

$$\triangle w = w_{n+1} - w_n = \frac{(n + 1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L} \Rightarrow \frac{1}{L} = \frac{\triangle w}{\pi}$$
11.7 Fourier Integral: From Fourier Series to Fourier Integral

• We may write the Fourier series in this form

\[ f_L(x) = \frac{1}{2L} \int_{-L}^{L} f_L(v) dv + \frac{1}{\pi} \sum_{n=1}^{\infty} \left[ \cos w_n \Delta x \int_{-L}^{L} f_L(v) \cos w_n v dv + \sin w_n x \Delta w \int_{-L}^{L} f_L(v) \sin w_n v dv \right] \]

• Let \( L \to \infty \) and assume that the resulting nonperiodic function \( f(x) = \lim_{L \to \infty} f_L(x) \) is absolutely integrable on the x-axis
  - \( f_L(x) \) is absolutely integrable if the following limit exists

\[ \lim_{a \to -\infty} \int_{a}^{0} |f(x)| dx + \lim_{b \to \infty} \int_{0}^{b} |f(x)| dx \]

• Since \( 1/L \to 0 \) and \( \int_{-\infty}^{\infty} f_L(v) dv \) has a finite value,

\[ \lim_{L \to 0} \frac{1}{2L} \int_{-L}^{L} f_L(v) dv \to 0 \]

Note. Absolutely integrable: integral of the absolute value over the whole domain is finite.
11.7 Fourier Integral: From Fourier Series to Fourier Integral

- Also $\triangle w = \pi / L \to 0$

$$ f_L(x) = \frac{1}{\pi} \int_0^\infty \left[ \cos wx \int_{-\infty}^\infty f(v) \cos wvdv + \sin wx \int_{-\infty}^\infty f(v) \sin wvdv \right] dw $$

- If we introduce the notations

$$ A(w) = \frac{1}{\pi} \int_{-\infty}^\infty f(v) \cos wvdv $$

$$ B(w) = \frac{1}{\pi} \int_{-\infty}^\infty f(v) \sin wvdv $$

A representation of $f(x)$ by a Fourier Integral is

$$ f(x) = \int_0^\infty [A(w) \cos wx + B(w) \sin wx] dw $$
11.7 Fourier Integral: From Fourier Series to Fourier Integral

Theorem 37 (Sec. 11.7 Theorem 1, Fourier Integral)

If $f(x)$ is piecewise continuous in every finite interval and has a right-hand derivative and a left-hand derivative at every point and if the $f(x)$ is \textbf{absolutely integrable}, then $f(x)$ can be represented by a Fourier integral

$$f(x) = \int_{0}^{\infty} [A(w) \cos wx + B(w) \sin wx] dw$$

with $A$ and $B$ given by

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wvdv$$

$$B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wvdv$$

At a point where $f(x)$ is discontinuous the value of the Fourier integral equals the average of the left- and right-hand limits of $f(x)$ at that point.
11.7 Fourier Integral: Applications of Fourier Integrals

- Applications of Fourier Integrals:
  - Solving ODEs and PDEs
  - Use Fourier integrals in integration and in discussing functions defined by integrals

Example 38 (Sec. 11.7 Example 2, Single Pulse, Sine Integral. Dirichlet’s Discontinuous Factor. Gibbs Phenomenon)

Find the Fourier Integral representation of the function

\[ f(x) = \begin{cases} 
1, & |x| < 1 \\
0, & |x| > 1 
\end{cases} \]  (13)
11.7 Fourier Integral: Example 2’s Answer

\[ A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wvdv = \frac{1}{\pi} \int_{-1}^{1} \cos wvdv = \frac{1}{\pi} \frac{\sin wv}{w} \bigg|_{-1}^{1} = \frac{2 \sin w}{w\pi} \]

\[ B(w) = \frac{1}{\pi} \int_{-1}^{1} \sin wvdv = \frac{1}{\pi} \left( -\frac{\cos wv}{w} \right) \bigg|_{-1}^{1} = 0 \]

\[ \Rightarrow f(x) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\cos wx \sin w}{w} dw \]

Notice that the average of the left- and right-hand limits of \( f(x) \) at \( x = 1 \) is equal to 
\( (1 + 0)/2 = 1/2 \),

\[ \int_{0}^{\infty} \frac{\cos wx \sin w}{w} dw = \begin{cases} \pi/2, & 0 \leq x \leq 1 \\ \pi/4, & x = 1 \\ 0, & x > 1 \end{cases} \] [Dirichlet’s discontinuous factor]

If \( x = 0 \), \( \int_{0}^{\infty} \frac{\sin w}{w} dw = \pi/2 \)
11.7 Fourier Integral: Gibbs Phenomenon

- Define Sine integral

\[ S_i(u) = \int_0^u \frac{\sin w}{w} dw \]

- In the case of a Fourier series the graphs of the partial sums are approximation curves of the curve of the periodic function represented by the series.

- Similarly, in the case of the Fourier integral, approximations are obtained by replacing \( \infty \) by numbers \( a \). Hence, the integral \( \int_0^a \frac{\cos wx \sin w}{w} dw \) approximates \( f(x) \).
11.7 Fourier Integral: Gibbs Phenomenon

Figure 26: The integral for $a = 8, 16, \text{ and } 32$, illustrating the development of the Gibbs phenomenon

- **Gibbs phenomenon**: we might expect that these oscillations disappear as $a \to \infty$. But this is not true; with increasing $a$, they are shifted closer to the points $x = \pm 1$. This unexpected behavior, which also occurs in connection with Fourier series

\[
\frac{2}{\pi} \int_{0}^{a} \frac{\cos wx \sin w}{w} dw = \frac{1}{\pi} \int_{0}^{a} \frac{\sin(w + wx)}{w} dw + \frac{1}{\pi} \int_{0}^{a} \frac{\sin(w - wx)}{w} dw
\]

\[
= \frac{1}{\pi} \int_{0}^{(x+1)a} \frac{\sin t}{t} dt - \frac{1}{\pi} \int_{0}^{(x-1)a} \frac{\sin t}{t} dt
\]

\[
= \frac{1}{\pi} S_i(a[x + 1]) - \frac{1}{\pi} S_i(a[x - 1])
\]

[Note: let the first integral $(w + wx) = t$; let the last integral $(w - wx) = -t;$]
11.7 Fourier Integral: Fourier Cosine Integral and Fourier Sine Integral

Simplifications:

- If $f$ has a Fourier integral representation and is even, then $B(w) = 0$, the Fourier integral reduces to a **Fourier cosine integral**,

  \[ f(x) = \int_0^\infty A(w) \cos wx \, dw \]

  where

  \[ A(w) = \frac{2}{\pi} \int_0^\infty f(v) \cos wv \, dv \]

- If $f$ has a Fourier integral representation and is odd, then $A(w) = 0$, the Fourier integral reduces to a **Fourier sine integral**,

  \[ f(x) = \int_0^\infty B(w) \sin wx \, dw \]

  where

  \[ B(w) = \frac{2}{\pi} \int_0^\infty f(v) \sin wv \, dv \]
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11.8 Fourier Cosine and Sine Transforms

- An **integral transform** is a transformation in the form of an integral that produces from given functions to new functions depending on a different variable.
  - As tools in solving ODEs, PDEs, and integral equations and can often be of help in handling and applying special functions.
- **Laplace transform**: by far the most important integral transform in engineering.
- **Fourier transform**: can be obtained from the Fourier integral in a straightforward way.

\[
\begin{align*}
\Rightarrow \quad & \begin{cases} 
11.8 \text{ Fourier Cosine/Sine Transforms: Real Value} \\
11.9 \text{ Fourier Transform: Complex Value}
\end{cases}
\end{align*}
\]
11.8 Fourier Cosine and Sine Transforms

- The Fourier cosine transform concerns even function $f(x)$. We obtain it from the Fourier cosine integral:

$$f(x) = \int_{0}^{\infty} A(w) \cos wx \, dw \quad \text{where} \quad A(w) = \frac{2}{\pi} \int_{0}^{\infty} f(v) \cos wv \, dv$$

- Now we set $A(w) = \sqrt[4]{\frac{2}{\pi}} \hat{f}_c(w)$, then writing $v = x$ in the formula of $A(w)$,

**Fourier Cosine Transform:**

$$\hat{f}_c(w) = \sqrt[4]{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos wx \, dx$$

**Inverse Fourier Cosine Transform:**

$$f(x) = \sqrt[4]{\frac{2}{\pi}} \int_{0}^{\infty} \hat{f}_c(w) \cos wx \, dw$$

- We changed the notation $A$ to get a symmetric distribution of the constant $2/\pi$ in the original formula. This redistribution ($\sqrt[4]{2/\pi}$ each) is a standard convenience, but not essential.

- Other notations of Fourier Cosine Transform/Inverse Fourier Cosine Transform are $\mathcal{F}_c(f) = \hat{f}_c$ and $\mathcal{F}_c^{-1}(\hat{f}_c) = f$
11.8 Fourier Cosine and Sine Transforms

- The Fourier sine transform concerns odd function $f(x)$. We obtain it from the Fourier sine integral:

$$f(x) = \int_0^{\infty} B(w) \sin wx \, dw \text{ where } B(w) = \frac{2}{\pi} \int_0^{\infty} f(v) \sin vw \, dv$$

- Now we set $B(w) = \sqrt{\frac{2}{\pi}} \hat{f}_s(w)$, then writing $v = x$ in the formula of $B(w)$,

**Fourier Sine Transform:** $\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin wx \, dx$

**Inverse Fourier Sine Transform:** $f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_s(w) \sin wx \, dw$

- Other notations of Fourier Sine Transform/Inverse Fourier Sine Transform are $\mathcal{F}_s(f) = \hat{f}_s$ and $\mathcal{F}_s^{-1}(\hat{f}_s) = f$
11.8 Fourier Cosine and Sine Transforms

Example 39 (Sec. 11.8, Example 1, Fourier Cosine and Fourier Sine Transforms)

Find the Fourier cosine and Fourier sine transforms of the function

\[ f(x) = \begin{cases} 
  k, & 0 < x < a \\
  0, & x > a 
\end{cases} \]

\[ \hat{f}_c(w) = \sqrt{\frac{2}{\pi}} k \int_0^a \cos wx \, dx = \sqrt{\frac{2}{\pi}} k \left( \frac{\sin aw}{w} \right) \]

\[ \hat{f}_s(w) = \sqrt{\frac{2}{\pi}} k \int_0^a \sin wx \, dx = \sqrt{\frac{2}{\pi}} k \left( 1 - \frac{\cos aw}{w} \right) \]

- **Note:** When \( f(x) \) is a constant function (\( f(x) = 0 \) for \( x > 0 \)), these transforms do not exist. Why?
  - **Absolutely integrable** is a sufficient condition of Fourier integral
Example 40 (Sec. 11.8, Example 2, Fourier Cosine Transform of the Exponential Function)

Find $F_c(e^{-x})$

By integration by parts and recursion,

$$F_c(e^{-x}) = \frac{\sqrt{2}}{\pi} \int_0^\infty e^{-x} \cos wx \, dx$$

$$= \left[ \frac{\sqrt{2} e^{-x} \sin wx}{\pi w} \right]_0^\infty + \frac{\sqrt{2}}{\pi} \int_0^\infty e^{-x} \frac{\sin wx}{w} \, dx$$

$x \to \infty, e^{-x} = 0; x = 0, e^{-x} = 1, \sin wx = 0$

$$= 0 + \left[ \frac{2 e^{-x}}{\pi} \left( -\frac{\cos wx}{w^2} \right) \right]_0^\infty - \sqrt{\frac{2}{\pi}} \frac{1}{w^2} \int_0^\infty e^{-x} \cos wx \, dx$$

$x \to \infty, e^{-x} = 0; x = 0, e^{-x} = 1, \cos wx = 1$
11.8 Fourier Cosine and Sine Transforms

\[
\Rightarrow \left(1 + \frac{1}{w^2}\right) \int_0^\infty e^{-x} \cos wx \, dx = \sqrt{\frac{2}{\pi}} \frac{1}{w^2}
\]

\[
\Rightarrow \mathcal{F}_c(e^{-x}) = \sqrt{\frac{2}{\pi}} \frac{1}{1 + w^2}
\]

Note: You can also try to solve this problem by finding the Laplace transform of \(\cos wx\).
11.8 Fourier Cosine/Sine Transforms: Linearity, Transforms of Derivatives

**Theorem 41 (Existence of Fourier Cosine/Sine Transform)**

If \( f(x) \) is absolutely integrable on the positive x-axis and piecewise continuous on every finite interval, then the Fourier cosine and sine transforms of \( f \) exist.

**Theorem 42 (Linearity of Fourier Cosine/Sine Transform)**

If \( f \) and \( g \) have Fourier cosine and sine transforms, so does \( af + bg \) for any constants \( a \) and \( b \),

\[
\mathcal{F}_c(af + bg) = a\mathcal{F}_c(f) + b\mathcal{F}_c(g)
\]

\[
\mathcal{F}_s(af + bg) = a\mathcal{F}_s(f) + b\mathcal{F}_s(g)
\]

\[
\mathcal{F}_c(af + bg) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} [af(x) + bg(x)] \cos wx \, dx
\]

\[
= a\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos wx \, dx + b\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} g(x) \cos wx \, dx
\]
11.8 Fourier Cosine/Sine Transforms: Linearity, Transforms of Derivatives

**Theorem 43 (Cosine and Sine Transforms of Derivatives)**

Let $f(x)$ be continuous and absolutely integrable on the x-axis, let $f'(x)$ be piecewise continuous on every finite interval, and let $f(x) \to 0$ as $x \to \infty$. Then

$$
\mathcal{F}_c \{ f'(x) \} = w \mathcal{F}_s \{ f(x) \} - \sqrt{\frac{2}{\pi}} f(0)
$$

$$
\mathcal{F}_s \{ f'(x) \} = -w \mathcal{F}_c \{ f(x) \}
$$

\[
\mathcal{F}_c \{ f'(x) \} = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos wx \, dx
\]

\[
= \sqrt{\frac{2}{\pi}} \left[ f(x) \cos wx \bigg|_0^\infty + w \int_0^\infty f(x) \sin wx \, dx \right]
\]

\[
= -\sqrt{\frac{2}{\pi}} f(0) + w \mathcal{F}_s \{ f(x) \}
\]
11.8 Fourier Cosine/Sine Transforms: Linearity, Transforms of Derivatives

\[ \mathcal{F}_s\{f'(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \sin wx \, dx \]

\[ = \sqrt{\frac{2}{\pi}} \left[ f(x) \sin wx \bigg|_0^\infty - w \int_0^\infty f(x) \cos wx \, dx \right] \]

\[ = 0 - w \mathcal{F}_c\{f(x)\} \]
Theorem 44 (Cosine and Sine Transforms of Derivatives)

Let $f'(x)$ be continuous and absolutely integrable on the $x$-axis, let $f''(x)$ be piecewise continuous on every finite interval, and let $f(x) \to 0$ and $f'(x) \to 0$ as $x \to \infty$. Then

$$\mathcal{F}_c\{f''(x)\} = -w^2 \mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}}f'(0)$$

$$\mathcal{F}_s\{f''(x)\} = -w^2 \mathcal{F}_s\{f(x)\} + \sqrt{\frac{2}{\pi}}wf(0)$$
11.8 Fourier Cosine/Sine Transforms: Example 3

Example 45 (Sec. 11.8, Example 3)

Find the Fourier cosine transform \( \mathcal{F}_c(e^{-ax}) \) of \( f(x) = e^{-ax} \), where \( a > 0 \)

By differentiation, \( (e^{-ax})'' = a^2 e^{-ax} \)

\[
\Rightarrow a^2 f(x) = f''(x)
\]

\[
\Rightarrow a^2 \mathcal{F}_c(f) = \mathcal{F}_c(f'')
\]

\[
= -w^2 \mathcal{F}_c(f) - \sqrt{\frac{2}{\pi}} f'(0)
\]

\[
= -w^2 \mathcal{F}_c(f) + a\sqrt{\frac{2}{\pi}}
\]

Hence, \( (a^2 + w^2) \mathcal{F}_c(f) = a\sqrt{2/\pi} \)

\[
\Rightarrow \mathcal{F}_c(e^{-ax}) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + w^2}
\]
Outline

1. Preview Exercise
2. Overview of Chapter 11
3. Sec. 11.1 Fourier Series
4. Sec. 11.2 Arbitrary Period. Even and Odd Functions. Half-Range Expansions
5. Sec. 11.3 Forced Oscillations
6. Sec. 11.4 Approximation by Trigonometric Polynomials
7. Sec. 11.5 Sturm–Liouville Problems. Orthogonal Functions
8. Sec. 11.6 Orthogonal Series. Generalized Fourier Series
9. Sec. 11.7 Fourier Integral
10. Sec. 11.8 Fourier Cosine and Sine Transforms
11. Sec. 11.9 Fourier Transform. Discrete and Fast Fourier Transforms
11.9 Fourier Transform: Complex Form

- Complex Form of the Fourier Integral:

\[ f(x) = \int_{0}^{\infty} [A(w) \cos wx + B(w) \sin wx] dw \]

where

\[ A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv dv \]
\[ B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv dv \]

- Substituting A and B into the integral for f, we have,

\[ f(x) = \frac{1}{\pi} \int_{0}^{\infty} \left[ \int_{-\infty}^{\infty} f(v) \cos (wx - wv) dv \right] dw = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(v) \cos (wx - wv) dv \right] dw \]

\( F(w) \) is even function
11.9 Fourier Transform: Complex Form

- We claim that the integral of the form in the previous page with sin instead of cos is zero:

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(v) \sin(wx - wv) dv \right] dw = 0
\]  
(15)

- By Euler formula: \( e^{ix} = \cos x + i \sin x \):

\[
f(v) \cos(wx - wv) + i \cdot f(v) \sin(wx - wv) = f(v)e^{i(wx-wv)}
\]

- Hence, the result of adding (14) and \( i \) times (15):

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(v)e^{iw(x-v)} dv dw
\]  
(16)

- Writing the exponential function in (16) as a product of exponential functions

\[
f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v)e^{-iwv} dv \right] e^{iwx} dw
\]
11.9 Fourier Transform: Complex Form

• Writing the exponential function in (16) as a product of exponential functions

\[ f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v)e^{-i\omega v} \, dv \right] e^{i\omega x} \, d\omega \]

• Fourier transform of \( f \): the expression in brackets is a function of \( \omega \), is denoted by \( \hat{f}(\omega) \)

\[ \hat{f}(\omega) = \mathcal{F}(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\omega x} \, dx \]

• inverse Fourier transform of \( \hat{f}(\omega) \):

\[ f(x) = \mathcal{F}^{-1}(\hat{f}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x} \, d\omega \]
11.9 Fourier Transform: Complex Form

**Theorem 46 (Existence of the Fourier Transform)**

If \( f(x) \) is **absolutely integrable** on the \( x \)-axis and **piecewise continuous** on every finite interval, then the Fourier transform \( \hat{f}(w) \) of \( f(x) \) exists.

- Besides the definition in Advanced Engineering Mathematics, there are other definitions of FT

\[
\begin{align*}
\hat{f}(w) &= \int_{-\infty}^{\infty} f(x)e^{-iwx} \, dx \\
f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(w)e^{iwx} \, dw \\
\hat{f}(f) &= \int_{-\infty}^{\infty} f(x)e^{-i2\pi fx} \, dx \\
f(x) &= \int_{-\infty}^{\infty} \hat{f}(f)e^{i2\pi fx} \, df
\end{align*}
\]
Example 47 (Sec. 11.9, Example 1)

Find the Fourier transform of \( f(x) = 1 \) if \(|x| < 1\) and \( f(x) = 0 \) otherwise.

\[
\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \left. \frac{e^{-iwx}}{-iw} \right|_{-1}^{1} = \frac{1}{-iw\sqrt{2\pi}} (e^{-iw} - e^{iw})
\]

Since \( e^{iw} = \cos w + i \sin w \), we have \( e^{-iw} - e^{iw} = -2i \sin w \),

\[
\hat{f}(w) = \sqrt{\frac{2}{\pi}} \frac{\sin w}{w} = \sqrt{\frac{2}{\pi}} \text{sinc} \left( \frac{w}{\pi} \right)
\]
Example 48 (Sec. 11.9, Example 2)

Find the Fourier transform \( \hat{f}(w) \) of

\[
f(x) = \begin{cases} 
  e^{-ax} & \text{if } x > 0 \\
  0 & \text{otherwise}
\end{cases}
\]

\[
\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-ax} e^{-iwx} \, dx 
\]

\[
= \left. \frac{1}{\sqrt{2\pi}} \frac{e^{-(a+iw)x}}{-(a + iw)} \right|_{0}^{\infty} 
\]

\[
= \frac{1}{\sqrt{2\pi}(a + iw)} 
\]

Note: Compare the result with Example 3 of Sec. 11.8 — \( \mathcal{F}_c(e^{-ax}) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + w^2} \)
Theorem 49 (Linearity of the Fourier Transform)

The Fourier transform is a linear operation; that is, for any functions \( f(x) \) and \( g(x) \) whose Fourier transforms exist and any constants \( a \) and \( b \), the Fourier transform of \( af + bg \) exists, and \( \mathcal{F}\{af + bg\} = a\mathcal{F}\{f\} + b\mathcal{F}\{g\} \)

\[
\mathcal{F}\{af(x) + bg(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af(x) + bg(x)]e^{-iwx} dx
\]

\[
= a \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-iwx} dx + b \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)e^{-iwx} dx
\]

\[
= a\mathcal{F}\{f(x)\} + b\mathcal{F}\{g(x)\}
\]
Theorem 50 (Fourier Transform of the Derivative of $f(x)$)

Let $f(x)$ be continuous on the x-axis and $f(x) \to 0$ as $|x| \to \infty$. Furthermore, let $f'(x)$ be absolutely integrable on the x-axis. Then $\mathcal{F}\{f'(x)\} = iw\mathcal{F}\{f(x)\}$

\[
\mathcal{F}\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x)e^{-iwx} \, dx
\]

\[
= \frac{1}{\sqrt{2\pi}} f(x)e^{-iwx} \bigg|_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}} (-iw) \int_{-\infty}^{\infty} f(x)e^{-iwx} \, dx
\]

Since $f(x) \to 0$ as $|x| \to \infty$,

\[
\mathcal{F}\{f'(x)\} = iw\mathcal{F}\{f(x)\}
\]

Remark 51

\[
\mathcal{F}\{f''(x)\} = iw\mathcal{F}\{f'(x)\} = (iw)^2\mathcal{F}\{f(x)\} = -w^2\mathcal{F}\{f(x)\}
\]
Example 52 (Sec. 11.9, Example 3)

Find the Fourier transform of $xe^{-x^2}$ from the Fourier Transform Table

Note that $\mathcal{F}\{e^{-x^2}\} = \frac{1}{\sqrt{2}} e^{-\frac{w^2}{4}}$

\[
\mathcal{F}\{xe^{-x^2}\} = \mathcal{F}\left\{-\frac{1}{2} \left( e^{-x^2} \right)' \right\} = -\frac{1}{2} \mathcal{F}\{ (e^{-x^2})' \}
\]
\[
= -\frac{1}{2} iw \mathcal{F}\{ (e^{-x^2}) \}
\]
\[
= -\frac{1}{2} iw \frac{1}{\sqrt{2}} e^{-\frac{w^2}{4}}
\]
\[
= -\frac{iw}{2\sqrt{2}} e^{-\frac{w^2}{4}}
\]
11.9 Fourier Transform: Convolution Theorem

• The convolution $f \ast g$ of functions $f$ and $g$ is defined by

$$h(x) = (f \ast g)(x) = \int_{-\infty}^{\infty} f(p)g(x-p)\,dp = \int_{-\infty}^{\infty} f(x-p)g(p)\,dp$$

• The same purpose as Laplace transforms: taking the convolution of two functions and then taking the transform of the convolution is the same as multiplying the transforms of these functions (and multiplying them by $2\pi$)
Theorem 53 (Convolution Theorem)

Suppose that \( f(x) \) and \( g(x) \) are piecewise continuous, bounded, and absolutely integrable on the x-axis. Then

\[
\mathcal{F}\{f \ast g\} = \sqrt{2\pi} \mathcal{F}\{f\} \cdot \mathcal{F}\{g\}
\]

\[
\mathcal{F}\{f \ast g\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(p)g(x-p)dp \right) e^{-iwx} dx
\]

Instead of \( x \), we take a new variable \( q = x - p \). Therefore \( x = p + q \)

\[
\mathcal{F}\{f \ast g\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)g(q)e^{-iwp(p+q)} dp dq
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p)e^{-iwp} dp \int_{-\infty}^{\infty} g(q)e^{-iwq} dq
\]

\[
= \frac{1}{\sqrt{2\pi}} \left[ \sqrt{2\pi} \mathcal{F}\{f\} \right] \left[ \sqrt{2\pi} \mathcal{F}\{g\} \right]
\]

\[
= \sqrt{2\pi} \mathcal{F}\{f\} \cdot \mathcal{F}\{g\}
\]
A function $f(x)$ is given only in terms of values at finitely many points, and one is interested in extending Fourier analysis to this case.

Dealing with sampled values rather than with functions, we can replace the Fourier transform by the so-called discrete Fourier transform (DFT).

The main application of such a “discrete Fourier analysis” concerns large amounts of equally spaced data, as they occur in telecommunication, time series analysis, and various simulation problems.
11.9 Discrete Fourier Transform (DFT)

- Let \( f(x) \) be periodic, for simplicity of period \( 2\pi \). We assume that \( N \) measurements of \( f(x) \) are taken over the interval \( 0 \leq x \leq 2\pi \) at regularly spaced points

\[
x_k = \frac{2\pi k}{N} \quad \text{for} \quad k = 0, 1, \ldots, N - 1
\]

- We also say that \( f(x) \) is being sampled at these points. We now want to determine a complex trigonometric polynomial

\[
f_k = f(x_k) = q(x_k) = \sum_{n=0}^{N-1} c_ne^{inx_k}
\]

that interpolates \( f(x) \) at the nodes \( x_k \).
- Hence we must determine the coefficients \( c_0, c_1, \ldots, c_{N-1} \).
11.9 Discrete Fourier Transform (DFT)

- How to find \( c_n \)?
  - By an idea similar to that in Sec. 11.1 for deriving the Fourier coefficients by using the orthogonality of the trigonometric system
    - Instead of integrals we now take sums
  - We multiply both sides of (17) by \( e^{-imx_k} \) and sum over \( k \) from 0 to \( N - 1 \)

\[
\sum_{k=0}^{N-1} f_k e^{-imx_k} = \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} c_n e^{inx_k} e^{-imx_k}
\]

\[
= \sum_{n=0}^{N-1} c_n \left[ \sum_{k=0}^{N-1} e^{i(n-m)\frac{2\pi k}{N}} \right] = \sum_{n=0}^{N-1} c_n \sum_{k=0}^{N-1} \left[ e^{i(n-m)\frac{2\pi k}{N}} \right]^k
\]

- Let \( r = e^{i(n-m)\frac{2\pi}{N}} \), \( \sum_{k=0}^{N-1} \left[ e^{i(n-m)\frac{2\pi}{N}} \right]^k = \frac{r^N(1-r^N)}{1-r} \) when \( r \neq 1 \)
- When \( n \neq m \) \((r \neq 1)\), \( r^N = e^{i(n-m)\cdot2\pi k} = 1 \Rightarrow \sum_{k=0}^{N-1} r^k = 0 \)
- When \( n = m \Rightarrow r = 1 \Rightarrow \sum_{k=0}^{N-1} r^k = N \)

\[
\Rightarrow \sum_{k=0}^{N-1} \left[ e^{i(n-m)\frac{2\pi}{N}} \right]^k = \begin{cases} 
0, & n \neq m \\
N, & n = m 
\end{cases}
\]

- When \( n \neq m \) 
  - Let \( r = e^{i(n-m)\frac{2\pi}{N}} \)
  - \( \sum_{k=0}^{N-1} r^k = \frac{r^N(1-r^N)}{1-r} \) when \( r \neq 1 \)
  - \( r^N = e^{i(n-m)\cdot2\pi k} = 1 \Rightarrow \sum_{k=0}^{N-1} r^k = 0 \)
  - When \( n = m \Rightarrow r = 1 \Rightarrow \sum_{k=0}^{N-1} r^k = N \)

\[
\Rightarrow \sum_{k=0}^{N-1} \left[ e^{i(n-m)\frac{2\pi}{N}} \right]^k = \begin{cases} 
0, & n \neq m \\
N, & n = m 
\end{cases}
\]
11.9 Discrete Fourier Transform (DFT)

\[ \sum_{k=0}^{N-1} f_k e^{-im \frac{2\pi}{N} k} = c_m \cdot N \Rightarrow c_m = \frac{1}{N} \sum_{k=0}^{N-1} f_k e^{-im \frac{2\pi}{N} k} \]

- Since computation of the \( c_n \) (by the fast Fourier transform) involves successive halving of the problem size \( N \), it is practical to drop the factor \( \frac{1}{N} \) from \( c_n \)
- Discrete Fourier transform is defined by \( \hat{f}_n = Nc_n \)

**Definition 54 (DFT and IDFT)**

Define the discrete Fourier transform of the given signal \( f = [f_0 f_1 \ldots f_{N-1}]^T \) to be the vector \( \hat{f} = [\hat{f}_0 \hat{f}_1 \ldots \hat{f}_{N-1}]^T \) with components,

\[ \hat{f}_n = \sum_{k=0}^{N-1} f_k e^{-in \frac{2\pi}{N} k} \text{ where } n = 0 \ldots N-1 \]

The inverse discrete Fourier transform (IDFT) is defined as

\[ f_k = \frac{1}{N} \sum_{n=0}^{N-1} \hat{f}_n e^{in \frac{2\pi}{N} k} \text{ where } k = 0 \ldots N-1 \]
11.9 Discrete Fourier Transform (DFT)

Example 55 (Sec. 11.9, Example 4, DFT, Sample of $N = 4$ Values)

Let $N = 4$ measurements (sample values) be given. Let the sample values be, say $\mathbf{f} = [0 \ 1 \ 4 \ 9]^T$. Find the DFT of $\mathbf{f}$ to be $\hat{\mathbf{f}} = [\hat{f}_0, \hat{f}_1, \hat{f}_2, \hat{f}_3]^T$.

When $N = 4$, $\hat{f}_n = \sum_{k=0}^{N-1} f_k e^{-in \frac{2\pi}{4} k} = \sum_{k=0}^{N-1} f_k e^{-in \frac{2\pi}{2} k}$,

$$\hat{f}_0 = \sum_{k=0}^{N-1} f_k e^{-i0 \frac{\pi}{2} k} = \sum_{k=0}^{N-1} f_k \cdot 1 = 0 + 1 + 4 + 9 = 14$$

$$\hat{f}_1 = \sum_{k=0}^{N-1} f_k e^{-i1 \frac{\pi}{2} k} = 0e^{-0i} + 1e^{-i \frac{\pi}{2}} + 4e^{-i \frac{\pi}{2} 2} + 9e^{-i \frac{3\pi}{2}} = -i - 4 + 9i = -4 + 8i$$

$$\hat{f}_2 = \sum_{k=0}^{N-1} f_k e^{-i\pi k} = 0e^{-i0} + 1e^{-i\pi} + 4e^{-i\pi 2} + 9e^{-i\pi 3} = -1 + 4 - 9 = -6$$

$$\hat{f}_3 = \sum_{k=0}^{N-1} f_k e^{-i3 \frac{\pi}{2} k} = 0e^{-i0} + 1e^{-i3 \frac{\pi}{2}} + 4e^{-i3 \pi} + 9e^{-i9 \frac{\pi}{2}} = i - 4 - 9i = -4 - 8i$$
11.9 Discrete Fourier Transform (DFT)

Definition 56 (DFT and IDFT)

Define the DFT of the given signal \( f = [f_0, f_1, \ldots, f_{N-1}]^T \) to be the vector \( \hat{f} = [\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_{N-1}]^T \). In vector notation, the DFT

\[
\hat{f} = F_N f
\]

where the \( N \times N \) Fourier matrix \( F_N = [e_{nk}] \) has the entries

\[
e_{nk} = e^{-inx_k} = e^{-i \frac{2\pi}{N} nk} = w^{nk}, \quad \text{where } n, k = 0, \ldots, N - 1
\]

\( F_N \) and its complex conjugate \( \bar{F}_N = \frac{1}{N} [\bar{w}_{nk}] \) satisfy

\[
\bar{F}_N F_N = F_N \bar{F}_N = NI
\]

where \( I \) is the \( N \times N \) unit matrix. Hence \( F_N^{-1} = \frac{1}{N} \bar{F}_N \) and the IDFT is calculated as

\[
f = F_N^{-1} \hat{f} = \frac{1}{N} \bar{F}_N \hat{f}
\]
11.9 Discrete Fourier Transform (DFT)

Example 57 (Sec. 11.9, Example 4, DFT in matrix form, Sample of \( N = 4 \) Values)

Let \( N = 4 \) measurements (sample values) be given. Let the sample values be, say \( f = [0 \ 1 \ 4 \ 9]^T \). Find the DFT of \( f \) to be \( \hat{f} = [\hat{f}_0 \ \hat{f}_1 \ \hat{f}_2 \ \hat{f}_3]^T \).

When \( N = 4 \), \( w = e^{-i \frac{2\pi}{4}} = e^{-i \frac{\pi}{2}} = -i \) and \( w^{nk} = (-i)^{nk} \)

\[
\hat{f} = F_4 f = \begin{bmatrix}
w_0 & w_01 & w_02 & w_03 \\
w_{10} & w_{11} & w_{12} & w_{13} \\
w_{20} & w_{21} & w_{22} & w_{23} \\
w_{30} & w_{31} & w_{32} & w_{33}
\end{bmatrix}
f = \begin{bmatrix}
w_0 & w_0 & w_0 & w_0 \\
w_0 & w_1 & w_2 & w_3 \\
w_0 & w_2 & w_4 & w_6 \\
w_0 & w_3 & w_6 & w_9
\end{bmatrix}
f
\]

\[
= \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & -i & -1 & i \\
1 & -1 & 1 & -1 \\
1 & i & -1 & -i
\end{bmatrix}
\begin{bmatrix}
0 \\
1 \\
4 \\
9
\end{bmatrix}
= \begin{bmatrix}
14 \\
-4 + 8i \\
-6 \\
-4 - 8i
\end{bmatrix}
\]
11.9 Fast Fourier Transforms (FFT)

- The DFT requires $O(N^2)$ operations
- The FFT is a computational method for the DFT that needs only $O(N \times \log_2 N)$ operations
  - For $N = 1000$ sample points, those operations are reduced by a factor $1000/\log_2 1000 \approx 100$
  - Here one chooses $N = 2^p$ ( $p$ is an integer) and uses the special form of the Fourier matrix to break down the given problem into smaller problems
- The breakdown produces two problems of size $M = N/2$. This breakdown is possible because for $N = 2M$ we have
  \[ w_N^2 = w_{2M}^2 = (e^{-2\pi i/N})^2 = e^{-4\pi i/2M} = e^{-2\pi i/M} = w_M \]

  by defining $w_N^m = e^{-i2\pi m/N}$ and $w_N = e^{-i2\pi /N}$
11.9 Fast Fourier Transforms (FFT)

Definition 58 (Fast Fourier Transforms (FFT))

1. Split into two vectors with \( M \) components each:

\[
\begin{align*}
f_{ev} &= \left[f_0 \ f_2 \ \ldots \ f_{N-2}\right]^T \\
f_{od} &= \left[f_1 \ f_3 \ \ldots \ f_{N-1}\right]^T
\end{align*}
\]

2. For \( f_{ev} \) and \( f_{od} \) we determine the \( M \)-point DFTs, involving \( M \times M \) matrix \( F_M \)

\[
\begin{align*}
\hat{f}_{ev} &= \left[\hat{f}_{ev,0} \ \hat{f}_{ev,1} \ \ldots \ \hat{f}_{ev,M-1}\right]^T = F_M f_{ev} \\
\hat{f}_{od} &= \left[\hat{f}_{od,0} \ \hat{f}_{od,1} \ \ldots \ \hat{f}_{od,M-1}\right]^T = F_M f_{od}
\end{align*}
\]

3. From these vectors we obtain the components of the DFT of the given vector \( f \) by the formulas,

\[
\begin{align*}
\hat{f}_n &= \hat{f}_{ev,n} + w_N^n \cdot \hat{f}_{od,n} & n = 0, \ldots, M - 1 \\
\hat{f}_{n+M} &= \hat{f}_{ev,n} - w_N^n \cdot \hat{f}_{od,n} & n = 0, \ldots, M - 1
\end{align*}
\]
Find the DFT of $f = [f_0 \ f_1 \ f_2 \ f_3]^T$ to be $\hat{f} = [\hat{f}_0 \ \hat{f}_1 \ \hat{f}_2 \ \hat{f}_3]^T$ using FFT.

1. Split into two vectors with $M = N/2 = 2$ components: $f_{ev} = [f_0 \ f_2]^T$ and $f_{od} = [f_1 \ f_3]^T$

2. For $f_{ev}$ and $f_{od}$ we determine the 2-point DFTs ($w = e^{-2\pi i/2} = -1$)

   $$\hat{f}_{ev} = [\hat{f}_{ev,0} \ \hat{f}_{ev,1}]^T = F_2f_{ev} = \begin{bmatrix} w^0 & w^0 \\ w^0 & w^1 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} f_0 \\ f_2 \end{bmatrix} = \begin{bmatrix} f_0 + f_2 \\ f_2 - f_2 \end{bmatrix}$$

   $$\hat{f}_{od} = [\hat{f}_{od,0} \ \hat{f}_{od,1}]^T = F_2f_{od} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_3 \end{bmatrix} = \begin{bmatrix} f_1 + f_3 \\ f_1 - f_3 \end{bmatrix}$$

3. For $w_4^0 = e^{-i0} = 1$ and $w_4^1 = e^{-2\pi i/4} = -i$, the components of the DFT is obtained as

   $$\hat{f}_0 = \hat{f}_{ev,0} + w_4^0 \cdot \hat{f}_{od,0} = (f_0 + f_2) + (f_1 + f_3)$$

   $$\hat{f}_1 = \hat{f}_{ev,1} + w_4^1 \cdot \hat{f}_{od,1} = (f_0 - f_2) + (-i)(f_1 - f_3)$$

   $$\hat{f}_2 = \hat{f}_{ev,0} - w_4^0 \cdot \hat{f}_{od,0} = (f_0 + f_2) - (f_1 + f_3)$$

   $$\hat{f}_3 = \hat{f}_{ev,1} - w_4^1 \cdot \hat{f}_{od,1} = (f_0 - f_2) - (-i)(f_1 - f_3)$$

(18)
Question & Answer