Power Dividers and Directional Couplers (7)

The T-Junction Power Divider (7.2)

Lossless Divider

\[ Y_{\text{in}} = jB + \frac{1}{Z_1} + \frac{1}{Z_2} = \frac{1}{Z_0} \]

1. Lossless
2. Match at the input port.
3. Mismatch at the output ports.
4. No isolation at the output ports.

Resistive Divider

1. Lossy.
2. Match at all ports.
3. No isolation.
From the figure

\[ V = V_1 \left( \frac{2Z_0/3}{Z_0/3 + 2Z_0/3} \right) = \frac{2}{3} V_1 \]

\[ V_2 = V_3 = V \frac{Z_0}{Z_0 + Z_0/3} = \frac{3}{4} V = \frac{1}{2} V_1 \]

\[
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}
\]

\[ S = \frac{1}{2} \begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix} \]

**The Wilkinson Power Divider (7.3)**

1. Matched at all ports.
   \[ S_{11} = S_{22} = S_{33} = 0 \]
2. Isolation between output ports.
   \[ S_{23} = S_{32} = 0 \]
3. No power loss from input to output ports.
   \[ S_{12} = S_{21} = S_{13} = S_{31} = -\frac{j}{\sqrt{2}} \]
4. Half power loss from output to input ports.

**Analysis**

1. Excite port 1.
Symmetry → equal voltages at port 2 and 3 → no current flows through the resistor → open. The circuit becomes

From the figure

\[ Z_{\text{in}} = Z_0 \Rightarrow S_{11} = 0 \]

To compute \( S_{21} = 0 \), let \( V^+ \) and \( V^- \) denote the voltages of the forward and backward propagating modes in one of the two \( \frac{\lambda}{4} \) lines. Assume the reference plane is located at port 1. Let the voltage of the incident wave at port 1 be \( V_1 \) and port 3 \( V_3 \). We have at port 1

\[
\begin{align*}
    V_1 &= V^+ + V^- \\
    \frac{V_1}{2Z_0} &= \frac{V^+-V^-}{\sqrt{2}Z_0} \\
    V^+ &= \frac{\sqrt{2}+1}{2\sqrt{2}}V_1 \\
    V^- &= \frac{\sqrt{2}-1}{2\sqrt{2}}V_1
\end{align*}
\]

At port 3,

\[
V_3 = -V^+ j + V^- j = -\frac{\sqrt{2}+1}{2\sqrt{2}} V_1 j + \frac{\sqrt{2}-1}{2\sqrt{2}} V_1 j = -\frac{j}{\sqrt{2}} V_1 \Rightarrow S_{31} = -\frac{j}{\sqrt{2}}
\]

2. Even and odd mode excitation at port 2 and 3

Rearranging the circuit as follow
a. Even mode: Symmetry $\Rightarrow$ equal voltages at port 2 and 3 $\Rightarrow$ no current flows through the resistor $\Rightarrow$ open. The circuit becomes

$$Z_{in}^e = Z_0 \Rightarrow S_{ee} = S_{oe} = 0$$

b. Odd mode: Anti-symmetry $\Rightarrow$ opposite voltages at port 2 and 3 $\Rightarrow$ short at the middle of the resistor. The circuit becomes

$$Z_{in}^o = Z_0 \Rightarrow S_{eo} = S_{oo} = 0$$

Since
From $s_{ee}=s_{oo}=s_{oe}=0$, we have $s_{22}=s_{33}=s_{23}=s_{32}=0$

**Unequal Power Division**

\[
V_2^+ = \frac{V_e^+ + V_o^+}{2}, \quad V_3^+ = \frac{V_e^+ - V_o^+}{2}
\]
\[
V_2^- = \frac{V_e^- + V_o^-}{2}, \quad V_3^- = \frac{V_e^- - V_o^-}{2}
\]

If power ration between ports 2 and 3 is $k^2 = \frac{P_3}{P_2}$,

\[
Z_{03} = Z_0 \sqrt{1 + K^2} \over K^3
\]
\[
Z_{02} = K^2 Z_{03} = Z_0 \sqrt{K(1 + K^2)}
\]
\[
R = Z_0 \left( K + \frac{1}{K} \right)
\]

N-way, equal-split, Wilkinson power divider
Basic Properties of a Three Port Device (7.1)

Impossible scenario: reciprocal, matching at all ports, lossless.

Reciprocal and matching at all ports give the following $S$ matrix

$$
[S] = \begin{bmatrix}
0 & S_{12} & S_{13} \\
S_{12} & 0 & S_{23} \\
S_{13} & S_{23} & 0
\end{bmatrix}
$$

If lossless, the matrix is unitary, that is,

- $|S_{12}|^2 + |S_{13}|^2 = 1$
- $|S_{12}|^2 + |S_{23}|^2 = 1$
- $|S_{13}|^2 + |S_{23}|^2 = 1$
- $S_{13}^* S_{23} = 0$
- $S_{23}^* S_{12} = 0$
- $S_{12}^* S_{13} = 0$

Two of $(S_{12}, S_{13}, S_{23})$ must be zero to satisfy the last 3 equations. However, then, the first 3 equations will not be satisfied.

Possible scenario:

1. Nonreciprocal, matching at all ports, lossless.

$$
[S] = \begin{bmatrix}
0 & S_{21} & S_{31} \\
S_{12} & 0 & S_{32} \\
S_{13} & S_{23} & 0
\end{bmatrix}
$$

Lossless
Two possible solutions
\[ S_{12} = S_{23} = S_{31} = 0, \quad |S_{21}| = |S_{32}| = |S_{13}| = 1 \]
and
\[ S_{21} = S_{32} = S_{13} = 0, \quad |S_{12}| = |S_{23}| = |S_{31}| = 1 \]

Example: Circulators

2. Reciprocal, lossless, matching only two ports.

\[
[S] = \begin{bmatrix}
0 & S_{21} & S_{31} \\
S_{12} & 0 & S_{32} \\
S_{13} & S_{23} & S_{33}
\end{bmatrix}
\]

Lossless
\[ |S_{12}|^2 + |S_{13}|^2 = 1 \]
\[ |S_{12}|^2 + |S_{23}|^2 = 1 \]
\[ |S_{13}|^2 + |S_{23}|^2 + |S_{33}|^2 = 1 \]
\[ S_{13}^* S_{23} = 0 \]
\[ S_{23} S_{12} + S_{33} S_{13} = 0 \]
\[ S_{12}^* S_{13} + S_{23}^* S_{33} = 0 \]

Possible solution
3. Lossy, matching at all ports, reciprocal.

**Basic Properties of a Four Port Device (7.1)**

Reciprocal, matched at all ports.

$$[S] = \begin{bmatrix} 0 & S_{12} & S_{13} & S_{14} \\ S_{12} & 0 & S_{23} & S_{24} \\ S_{13} & S_{23} & 0 & S_{34} \\ S_{14} & S_{24} & S_{34} & 0 \end{bmatrix}$$

If lossless and $S_{14} = S_{23} = 0$ (directional coupler), the following conditions are required.

$$S_{12} = S_{34} = \alpha$$
$$S_{13} = \beta e^{i\theta}$$
$$S_{24} = \beta e^{i\phi}$$
$$\theta + \phi = \pi \pm 2n\pi$$
$$\alpha^2 + \beta^2 = 1$$

where $\alpha, \beta, \theta, \phi$ are real.
1. Symmetrical: \( \theta = \varphi = \frac{\pi}{2} \)

\[
[S] = \begin{bmatrix}
0 & \alpha & j\beta & 0 \\
\alpha & 0 & 0 & j\beta \\
j\beta & 0 & 0 & \alpha \\
0 & j\beta & \alpha & 0
\end{bmatrix}
\]

2. Anti-symmetrical: \( \theta = 0, \ \varphi = \pi \),

\[
[S] = \begin{bmatrix}
0 & \alpha & \beta & 0 \\
\alpha & 0 & 0 & -\beta \\
\beta & 0 & 0 & \alpha \\
0 & -\beta & \alpha & 0
\end{bmatrix}
\]

**Coupling** = \( C = 10 \log \frac{P_1}{P_3} = -20 \log \beta \) dB

**Directivity** = \( D = 10 \log \frac{P_3}{P_4} = 20 \log \frac{\beta}{|S_{14}|} \) dB

**Isolation** = \( I = 10 \log \frac{P_1}{P_4} = -20 \log |S_{14}| \) dB

**Insertion loss** = \( L = 10 \log \frac{P_1}{P_2} = -20 \log |S_{12}| \) dB
Detailed Formulation

1. Row $1^* \times Row 2$, Row $3 \times Row 4^*$
   
   $S_{13}^*S_{23} + S_{14}^*S_{24} = 0$
   
   $S_{14}^*S_{13} + S_{24}^*S_{23} = 0$

2. Multiply 1 with $S_{24}^*$ and $S_{13}^*$, respectively and then subtract
   
   $S_{14}^*(|S_{13}|^2 - |S_{24}|^2) = 0$

3. Similarly, Row $1^* \times Row 3$, Row $2 \times Row 4^*$
   
   $S_{12}^*S_{23} + S_{14}^*S_{34} = 0$
   
   $S_{14}^*S_{12} + S_{34}^*S_{23} = 0$

4. Multiply 3 with $S_{12}^*$ and $S_{34}^*$, respectively and then subtract
   
   $S_{23}(|S_{12}|^2 - |S_{34}|^2) = 0$

5. If $S_{14} = S_{23} = 0$,
   
   $|S_{12}|^2 + |S_{13}|^2 = 0$, $|S_{12}|^2 + |S_{24}|^2 = 0$, $|S_{13}|^2 + |S_{34}|^2 = 0$, $|S_{24}|^2 + |S_{34}|^2 = 0$
   
   $|S_{13}|^2 = |S_{24}|^2$, $|S_{12}|^2 = |S_{34}|^2$

6. Without lose generality, choose $S_{12} = S_{34} = \alpha$ and
   
   $S_{13} = \beta e^{j\theta}$, $S_{24} = \beta e^{j\phi}$, then Row $2^* \times Row 3$
   
   $S_{12}^*S_{13} + S_{24}^*S_{23} = \alpha \beta e^{j\theta} + \alpha \beta e^{-j\phi} = 0 \Rightarrow \theta + \phi = \pi \pm 2\pi$
The Quadrature (90°) Hybrid (7.5)

\[
[S] = -\frac{1}{\sqrt{2}} \begin{bmatrix}
0 & j & 1 & 0 \\
j & 0 & 0 & 1 \\
1 & 0 & 0 & j \\
0 & 1 & j & 0
\end{bmatrix}
\]

Even-Odd Mode Analysis

Even Mode

Using ABCD matrix, we have

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}_e = \begin{bmatrix}
1 & 0 \\
j & 1
\end{bmatrix} \begin{bmatrix}
0 & \frac{j}{\sqrt{2}} \\
\frac{j}{\sqrt{2}} & 0
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
j & 1
\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix}
-1 & j \\
j & -1
\end{bmatrix}
\]
\[ \Gamma_e = S_{11}^e = \frac{A + B - C - D}{A + B + C + D} = 0 \]

\[ T_e = S_{21}^e = \frac{2}{A + B + C + D} = -\frac{1}{\sqrt{2}}(1 + j) \]

**Odd mode**

\[
\begin{bmatrix}
  A & B \\
  C & D
\end{bmatrix}_o = \begin{bmatrix}
  1 & 0 \\
  -j & 1
\end{bmatrix} \begin{bmatrix}
  0 & j \\
  \frac{j}{\sqrt{2}} & 0
\end{bmatrix} \begin{bmatrix}
  1 & 0 \\
  -j & 1
\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix}
  1 & j \\
  j & 1
\end{bmatrix}
\]

\[ \Gamma_o = S_{11}^o = \frac{A + B - C - D}{A + B + C + D} = 0 \]

\[ T_o = S_{21}^o = \frac{2}{A + B + C + D} = \frac{1}{\sqrt{2}}(1 - j) \]

\[ S_{11} = \frac{1}{2} \Gamma_e + \frac{1}{2} \Gamma_o = 0 \]

\[ S_{21} = \frac{1}{2} T_e + \frac{1}{2} T_o = -\frac{j}{\sqrt{2}} \]

\[ S_{31} = \frac{1}{2} T_e - \frac{1}{2} T_o = -\frac{1}{\sqrt{2}} \]

\[ S_{41} = \frac{1}{2} \Gamma_e - \frac{1}{2} \Gamma_o = 0 \]
Coupled Line Directional Coupler (7.6)

\[
\begin{align*}
\Gamma_e &= \frac{Z_0 + Z_0e j\tan\theta_e Z_0 - Z_0}{Z_0e + Z_0e j\tan\theta_e + Z_0} = \frac{j\tan\theta_e \left(Z_0^2 - Z_0^2\right)}{2Z_0e Z_0 + j(Z_0^2 + Z_0^2)\tan\theta_e} \\
\Gamma_o &= \frac{Z_0o + Z_0o j\tan\theta_o Z_0 - Z_0}{Z_0o + Z_0o j\tan\theta_o + Z_0} = \frac{j\tan\theta_o \left(Z_0^2 - Z_0^2\right)}{2Z_0o Z_0 + j(Z_0^2 + Z_0^2)\tan\theta_o}
\end{align*}
\]
\[ T_e = \frac{(1 + \Gamma_e)(1 + \Gamma'_e)}{e^{j\theta_e} + \Gamma'_e e^{-j\theta_e}} \]

\[ = \frac{2Z_{0e}Z_0 + j2\tan\theta_e Z_{0e}^2}{2Z_{0e}Z_0 + j(Z_{0e}^2 + Z_0^2)\tan\theta_e} \]

\[ = \frac{2Z_{0e}Z_0 + j(Z_{0e}^2 + Z_0^2)\tan\theta_e}{2Z_{0e}Z_0 + j2Z_{0e}Z_0\sin\theta_e} \]

\[ T_o = \frac{(1 + \Gamma_o)(1 + \Gamma'_o)}{e^{j\theta_o} + \Gamma'_o e^{-j\theta_o}} \]

\[ = \frac{Z_{0o}Z_0 + j\tan\theta_o Z_{0o}^2}{2Z_{0o}Z_0 + j(Z_{0o}^2 + Z_0^2)\tan\theta_o} \]

\[ = \frac{Z_{0o}Z_0 + j2Z_{0o}Z_0\sin\theta_o}{2Z_{0o}Z_0 + j(Z_{0o}^2 + Z_0^2)\tan\theta_o} \]

\[ S_{11} = \frac{\Gamma_e + \Gamma_o}{2} = \frac{1}{2} \left( \frac{j\tan\theta_e (Z_{0e}^2 - Z_0^2)}{2Z_{0e}Z_0 + j(Z_{0e}^2 + Z_0^2)\tan\theta_e} + \frac{j\tan\theta_o (Z_{0o}^2 - Z_0^2)}{2Z_{0o}Z_0 + j(Z_{0o}^2 + Z_0^2)\tan\theta_o} \right) \]

Assume \( \theta_e = \theta_o \) and \( Z_{0e}Z_{0o} = Z_0^2 \), we have
\[ S_{11} = \frac{\Gamma_e + \Gamma_o}{2} \]
\[ = \frac{1}{2} \left( \frac{j\tan\theta_e (Z_{0e}^2 - Z_0^2)}{2Z_{0e}Z_0 + j(Z_{0e}^2 + Z_0^2)\tan\theta_e} + \frac{j\tan\theta_o (Z_{0o}^2 - Z_0^2)}{2Z_{0o}Z_0 + j(Z_{0o}^2 + Z_0^2)\tan\theta_o} \right) \]
\[ = j\tan\theta \left( \frac{Z_0^2(Z_{0e} - Z_{0o})}{2Z_0^3 + jZ_0^2(Z_{0e} + Z_{0o})\tan\theta} + \frac{Z_0^2(Z_{0o} - Z_{0e})}{2Z_0^3 + jZ_0^2(Z_{0o} + Z_{0e})\tan\theta} \right) \]
\[ = 0 \]
\[ S_{31} = \frac{\Gamma_e - \Gamma_o}{2} \]
\[ = j\tan\theta \left( \frac{Z_0^2(Z_{0e} - Z_{0o})}{2Z_0^3 + jZ_0^2(Z_{0e} + Z_{0o})\tan\theta} - \frac{Z_0^2(Z_{0o} - Z_{0e})}{2Z_0^3 + jZ_0^2(Z_{0o} + Z_{0e})\tan\theta} \right) \]
\[ = j\tan\theta \left( \frac{Z_{0e} - Z_{0o}}{2Z_0 + j(Z_{0e} + Z_{0o})\tan\theta} - \frac{Z_{0o} - Z_{0e}}{2Z_0 + j(Z_{0o} + Z_{0e})\tan\theta} \right) \]
\[ = \frac{j\tan^2(Z_{0o} - Z_{0o})}{2Z_0 + j(Z_{0e} + Z_{0o})\tan\theta} \]
\[ = \frac{jC\tan\theta}{\sqrt{1 - C^2 + j\tan\theta}} \]

where \( C = \frac{Z_{0e} - Z_{0o}}{Z_{0e} + Z_{0o}} \).
\[ S_{21} = \frac{T_e + T_o}{2} \]

\[
= \frac{Z_{0e}Z_0 + j\tan\theta_e Z_{0e}^2}{2Z_{0e}Z_0 + j(Z_{0e}^2 + Z_{0o}^2)\tan\theta_e} + \frac{Z_{0o}Z_0 + j\tan\theta_o Z_{0o}^2}{2Z_{0o}Z_0 + j(Z_{0o}^2 + Z_{0o}^2)\tan\theta_o} \\
= \frac{Z_0 + j\tan\theta Z_{0e}}{2Z_0 + j(Z_{0e} + Z_{0o})\tan\theta} + \frac{Z_0 + j\tan\theta Z_{0o}}{2Z_0 + j(Z_{0o} + Z_{0o})\tan\theta} \\
= \frac{2Z_0}{\cos\theta(2Z_0 + j(Z_{0e} + Z_{0o})\tan\theta)} \\
= \frac{\sqrt{1 - C^2}}{\sqrt{1 - C^2} \cos\theta + j\sin\theta} \\
S_{41} = \frac{T_e - T_o}{2} = 0
\]

If \( \theta = \frac{\pi}{2} \), \( S_{31} = C \) and \( S_{21} = -j\sqrt{1 - C^2} \). Also

\[
Z_{0e} = Z_0 \sqrt{\frac{1 + C}{1 - C}} \\
Z_{0o} = Z_0 \sqrt{\frac{1 - C}{1 + C}} 
\]

For \( \theta_e \neq \theta_o \), choose the mid-band frequency such that \( \tan\theta_e = \tan\theta = -\tan\theta_o \)
\[ S_{11} = \frac{\Gamma_e + \Gamma_o}{2} \]
\[ = \frac{1}{2} \left( \frac{j\tan\theta (Z_0^2 - Z_{o0}^2)}{2Z_0 Z_o + j(Z_0^2 + Z_{o0}^2)\tan\theta} - \frac{j\tan\theta (Z_{o0}^2 - Z_0^2)}{2Z_0 Z_o - j(Z_{o0}^2 + Z_0^2)\tan\theta} \right) \]
\[ = \frac{j\tan\theta (Z_{o0} - Z_0) Z_0}{2} \frac{2Z_0}{4Z_0^2 + (Z_0 + Z_{o0})^2 \tan^2 \theta} \]
\[ = \frac{(Z_{o0} - Z_0) Z_0}{4Z_0^2 + (Z_0 + Z_{o0})^2 \tan^2 \theta} \]
\[ S_{31} = \frac{\Gamma_e - \Gamma_o}{2} \]
\[ = \frac{1}{2} \left( \frac{(Z_0 - Z_{o0})}{2Z_0 + j(Z_0 + Z_{o0})\tan\theta} + \frac{(Z_{o0} - Z_0)}{2Z_0 - j(Z_{o0} + Z_0)\tan\theta} \right) \]
\[ = \frac{j\tan\theta (Z_{o0} - Z_0) - j(Z_0 + Z_{o0})\tan\theta}{4Z_0^2 + (Z_0 + Z_{o0})^2 \tan^2 \theta} \]
\[ = \frac{(Z_0^2 - Z_{o0}^2) \tan^2 \theta}{4Z_0^2 + (Z_0 + Z_{o0})^2 \tan^2 \theta} \]
Example 7.7: Design a 20 dB single-section coupled line coupler in stripline with a ground plane spacing of 0.32 cm, a dielectric constant of 2.2, a characteristic impedance of 50 Ω, and a center frequency 3 GHz.

\[
C = 10^{-20/20} = 0.1
\]

\[
Z_{0e} = Z_0 \sqrt{\frac{1+C}{1-C}} = 55.28 \Omega
\]

\[
Z_{0o} = Z_0 \sqrt{\frac{1-C}{1+C}} = 45.23 \Omega
\]

![Graph showing coupling and directivity vs. frequency.](image)
Design of Multi-section Coupled Line Coupler

If \( C \ll 1 \),

\[
S_{31} = \frac{jC\tan\theta}{\sqrt{1 - C^2 + j\tan\theta}} \approx \frac{jC\tan\theta}{1 + j\tan\theta} = jC\sin\theta e^{-j\theta}
\]

\[
S_{21} = \frac{\sqrt{1 - C^2}}{\sqrt{1 - C^2 \cos\theta + j\sin\theta}} = e^{-j\theta}
\]

\[
S_{31} = jC_1\sin\theta e^{-j\theta} + jC_2\sin\theta e^{-2j\theta} + \cdots + jC_N\sin\theta e^{-2(N-1)\theta}
\]

If \( C_i = C_{N-i+1} \),

\[
S_{31} = j2C_1\sin\theta e^{-jN\theta} \left[ C_1\cos(N-1)\theta + C_2\cos(N-3)\theta + \cdots + \frac{1}{2}C_M \right]
\]

where \( M = \frac{N+1}{2} \)

Example 7.8: Design a three-section 20 dB coupled line coupler with a binomial response, a system impedance of 50 \( \Omega \), and a center frequency of 3 GHz.

\[
C = 2\sin\theta (C_1\cos2\theta + \frac{1}{2}C_2) = C_1\sin3\theta + (C_2 - C_1)\sin\theta
\]
\[ C\left(\frac{\pi}{2}\right) = C_2 - 2C_1 = 0.1 \]

\[ \frac{dC}{d\theta} = 3C_1 \cos 3\theta + (C_2 - C_1) \cos \theta \bigg|_{\pi/2} = 0 \]

\[ \frac{dC^2}{d\theta^2} = -9C_1 \sin 3\theta - (C_2 - C_1) \sin \theta \bigg|_{\pi/2} = 10C_1 - C_2 = 0 \]

\[ \therefore \quad C_1 = C_3 = 0.0125, \quad C_2 = 0.125 \]
180° Hybrid (7.8)

(a) Ring hybrid, rat-race.
(b) Tapered coupled line hybrid.
(c) Waveguide hybrid junction, magic-T.

\[
[S] = \frac{-j}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}
\]
The Lange Coupler

4-wire even mode:

\[ C_{e4} = C_{ex} + C_{in} \]

4-wire odd mode:

\[ C_{o4} = C_{ex} + C_{in} + 6C_{m} \]

2-wire even mode:

\[ C_{e} = C_{ex} \]

2-wire odd mode:

\[ C_{o} = C_{ex} + 2C_{m} \]

Also approximate \( C_{in} \) as follow:

\[ C_{in} = C_{ex} - \frac{C_{ex} C_{m}}{C_{ex} + C_{m}} \]
Then we have
\[ C_{e4} = \frac{C_e (3C_e + C_o)}{C_e + C_o} \]
\[ C_{o4} = \frac{C_o (3C_o + C_e)}{C_e + C_o} \]
\[ Z_{e4} = Z_0 e \frac{Z_0 e + Z_{0o}}{3Z_0 e + Z_{0o}} \]
\[ Z_{o4} = Z_{0o} \frac{Z_0 e + Z_{0o}}{3Z_{0o} + Z_0 e} \]

By applying coupled line theory
\[ Z_0 = \sqrt{Z_{e4} Z_{o4}} \]
and
\[ C = \frac{Z_{e4} - Z_{o4}}{Z_{e4} + Z_{o4}} \]

or in terms of \( Z_{0e} \) and \( Z_{0o} \),
\[ Z_{0e} = \frac{4C - 3 + \sqrt{9 - 8C^2}}{2C \sqrt{(1 - C)/(1 + C)}} Z_0 \]
\[ Z_{0o} = \frac{4C + 3 - \sqrt{9 - 8C^2}}{2C \sqrt{(1 + C)/(1 - C)}} Z_0 \]
Analysis of the Tapered Coupled Line Hybrid

Consider the taper as an ideal lossless transformer, the even mode ABCD matrix satisfy:

\[ V_1 = AV_2 + BI_2 \]
\[ I_1 = CV_2 + DI_2 \]

Since
\[ \frac{V_1}{I_1} = k \frac{V_2}{I_2}, \quad V_1 I_1 = V_2 I_2 \]

We have
\[
\frac{V_1}{V_2} = \sqrt{k}, \quad \frac{I_1}{I_2} = \frac{1}{\sqrt{k}}
\]

Therefore, the ABCD matrix of the transformers is
\[
\begin{bmatrix}
\sqrt{k} & 0 \\
0 & \frac{1}{\sqrt{k}}
\end{bmatrix}, \text{ for even mode.}
\]

Similarly,
\[
\begin{bmatrix}
\frac{1}{\sqrt{k}} & 0 \\
0 & \sqrt{k}
\end{bmatrix}, \text{ for odd mode.}
\]

Cascading all the ABCD matrices of the even mode, we have
\[
\begin{bmatrix}
\cos\theta & jZ_0 \sin\theta \\
jY_0 \sin\theta & \cos\theta
\end{bmatrix}\begin{bmatrix}
\frac{1}{\sqrt{k}} & 0 \\
0 & \sqrt{k}
\end{bmatrix}\begin{bmatrix}
\cos\theta & jZ_0 \sin\theta \\
jY_0 \sin\theta & \cos\theta
\end{bmatrix}
\]

\[
\begin{bmatrix}
\sqrt{k} \cos^2\theta - \frac{\sin^2\theta}{\sqrt{k}} & jZ_0 \left(\frac{1}{\sqrt{k}} + \sqrt{k}\right) \sin\theta \cos\theta \\
jY_0 \left(\frac{1}{\sqrt{k}} + \sqrt{k}\right) \sin\theta \cos\theta & \frac{\cos^2\theta}{\sqrt{k}} - k \sin^2\theta
\end{bmatrix}
\]

\[
S_{11}^e = \Gamma^e = \frac{k-1}{k+1}e^{-2j\theta} = -S_{22}^e
\]

\[
S_{21}^e = S_{12}^e = T^e = \frac{2\sqrt{k}}{k+1}e^{-2j\theta}
\]

Similarly, for odd mode
Thus,

\[ S_{11} = \Gamma = \frac{1-k}{k+1} e^{-2j\theta} = -S_{22} \]

\[ S_{21} = S_{12} = T = \frac{2\sqrt{k}}{k+1} e^{-2j\theta} \]

Thus,

\[ S_{44} = \frac{1}{2} (S_{11} + S_{11}) = 0 \]

\[ S_{24} = \frac{1}{2} (S_{11} - S_{11}) = \frac{k-1}{k+1} e^{-2j\theta} \]

\[ S_{34} = \frac{1}{2} (S_{21} + S_{21}) = \frac{2\sqrt{k}}{k+1} e^{-2j\theta} \]

\[ S_{14} = \frac{1}{2} (S_{21} - S_{21}) = 0 \]

\[ S_{11} = \frac{1}{2} (S_{22} + S_{22}) = 0 \]

\[ S_{31} = \frac{1}{2} (S_{11} - S_{11}) = \frac{1-k}{k+1} e^{-2j\theta} \]

Let \( \beta = |S_{34}|, \ \alpha = |S_{24}| \)

The S matrix of the tapered coupled is

\[
[S] = \begin{bmatrix}
0 & \beta & \alpha & 0 \\
\beta & 0 & 0 & -\alpha \\
\alpha & 0 & 0 & \beta \\
0 & -\alpha & \beta & 0
\end{bmatrix}
\]
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