Microwave Engineering

1. Microwave: 300MHz ~ 300 GHz, 1 m ~ 1mm.
   a. Not only apply in this frequency range. The real issue is wavelength. Historically, as early as WWII, this is the first frequency range we need to consider the wave effect.
   b. Why microwave engineering? We all know that the ideal of capacitor, inductor and resistance are first defined in DC.
      i. Circuit theory only apply in lower frequency.
      ii. Balance between two extreme: circuit and fullwave.

2. Today’s high tech is developed long time ago.
   a. Maxwell equation 1873. Theory before Experiment.
   b. Waveguide, Radar, Passive circuit, before WWII.

3. Transmission Line Theory
   a. Key difference between circuit theory and transmission line is electrical size.
      i. Ordinary circuit: no variation of current and voltage.
      ii. Transmission line: allow variation of current and voltage.
      iii. Definition of Voltage and current is
ambiguous in real waveguide except for TEM wave.
A Review of Electromagnetic Theories

Maxwell Equations (1873)

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]
\[ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \]
\[ \nabla \cdot \vec{D} = \rho \]
\[ \nabla \cdot \vec{B} = 0 \]

\( \vec{E}(x, y, z, t) \): Electric field intensity
\( \vec{D}(x, y, z, t) \): Electric flux density
\( \vec{H}(x, y, z, t) \): Magnetic field intensity
\( \vec{B}(x, y, z, t) \): Magnetic flux density
\( \vec{J}(x, y, z, t) \): Electric current density
\( \rho(x, y, z, t) \): Volume charge density

Constituent relationship
\[ \vec{D} = \varepsilon \vec{E} \]
\[ \vec{B} = \mu \vec{H} \]

where
\( \varepsilon \): Permittivity
\( \mu \): Permeability

Continuity relationship
\[ \nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \]
**Divergence Operator** \( \nabla \cdot \)

\[
\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}
\]

Physical meaning

\[
\nabla \cdot \vec{A} = \lim_{\Delta V \to 0} \frac{\phi_s}{\Delta V} \vec{A} \cdot d\vec{s}
\]

**Divergence Theorem**

\[
\int_V \nabla \cdot \vec{A} \, dv = \phi_s \vec{A} \cdot d\vec{s}
\]

**Curl Operator** \( \nabla \times \)

\[
\nabla \times \vec{A} = \begin{vmatrix}
\hat{a}_x & \hat{a}_y & \hat{a}_z \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
A_x & A_y & A_z
\end{vmatrix}
\]

Physical meaning

\[
\nabla \times \vec{A} \cdot \hat{a}_n = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_C \vec{A} \cdot d\vec{l}
\]

**Stoke’s theorem:**

\[
\int_S \nabla \times \vec{A} \cdot d\vec{s} = \oint_C \vec{A} \cdot d\vec{l}
\]
Time-Harmonic Fields

Time-harmonic:
\[ f(x,y,z,t) = f'(x,y,z)\cos(\omega t + \varphi) = \Re[f(x,y,z)'e^{j\varphi}e^{j\omega t}] = \Re[f(x,y,z)''e^{j\omega t}] \]

- \( f(x,y,z,t) \): a real function in both space and time.
- \( f(x,y,z)' \): a real function in space.
- \( f(x,y,z)'' \): a complex function in space. A phaser.

Thus, all derivative of time becomes
\[ \frac{\partial f(x,y,z,t)}{\partial t} = \Re[j\omega f(x,y,z)''e^{j\omega t}] \]

For a partial differential equation, all derivative of time can be replace with \( j\omega \), and all time dependence of \( e^{j\omega t} \) can be removed and becomes a partial differential equation of space only.

Representing all field quantities as
\[ \vec{E}(x,y,z,t) = \Re[\vec{E}(x,y,z)e^{j\omega t}] \]
\[ \vec{H}(x,y,z,t) = \Re[\vec{H}(x,y,z)e^{j\omega t}] \]
then the original Maxwell’s equation becomes
\[ \nabla \times \vec{E} = -j\omega \mu \vec{H} \]
\[ \nabla \times \vec{H} = -j\omega \varepsilon \vec{E} \]
\[ \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon} \]
\[ \nabla \cdot \vec{H} = 0 \]

Wave Equations
Source Free:
\[ \nabla \times \nabla \times \vec{E} = \omega^2 \mu \varepsilon \vec{E} \]
\[ \nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = \omega^2 \mu \varepsilon \vec{E} \quad (\nabla^2 = \nabla \cdot \nabla) \]
\[ \nabla^2 \vec{E} + \omega^2 \mu \varepsilon \vec{E} = 0 \]

**Plane Wave**
From wave equation
\[ \nabla^2 \vec{E} + k_0^2 \vec{E} = 0 \]

where \( k_0 = \omega \sqrt{\mu_0 \varepsilon_0} = \frac{\omega}{c} \), free space wave number or propagation constant.
In Cartesian coordinates, considering the x component,
\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) E_x + k_0^2 E_x = 0. \]
Assume \( E_x \) is independent of \( x \) and \( y \), then
\[ \frac{d^2 E_x}{dz^2} + k_0^2 E_x = 0. \]
The solutions are
\[ E_x = e^{\pm jk_0 z} \]
In time domain,
\[ E_x(z,t) = \Re \left[ e^{j(\omega t - k_0 z)} \right] = \cos(\omega t - k_0 z) \]
Constant phase \( \omega t - k_0 z = \text{constant} \)
\[ u_p = \frac{dz}{dt} = \frac{\omega}{k_0} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c \]
\[ \lambda_0 = \frac{2\pi}{k_0} \]
\[ \nabla \times \vec{E} = -j \omega \mu_0 \vec{H} \]
\[ H_y = \frac{E_x}{\eta_0} \]
\[ \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}, \text{ intrinsic impedance or wave impedance} \]

**Vector Potential**

\[ \nabla \cdot \vec{H} = 0 \Rightarrow \vec{H} = \nabla \times \vec{A} \]
\[ \vec{A} : \text{ vector potential} \]
\[ \nabla^2 \vec{A} + k^2 \vec{A} = -\vec{J} \]

**Flow of Electromagnetic Power and Poynting Vector**

\[ \nabla \cdot (\vec{E} \times \vec{H}) = H_y (\nabla \times \vec{E}) - E_z (\nabla \times \vec{H}) = -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} - \vec{E} \cdot \vec{J} \]

Since
\[ H_y \frac{\partial \vec{B}}{\partial t} = \vec{H} \cdot \frac{\partial (\mu \vec{H})}{\partial t} = \frac{1}{2} \frac{\partial (\mu \vec{H} \cdot \vec{H})}{\partial t} = \frac{\partial}{\partial t} \left( \frac{1}{2} \mu H^2 \right) \]
\[ E_z \frac{\partial \vec{D}}{\partial t} = \vec{E} \cdot \frac{\partial (\varepsilon \vec{E})}{\partial t} = \frac{1}{2} \frac{\partial (\varepsilon \vec{E} \cdot \vec{E})}{\partial t} = \frac{\partial}{\partial t} \left( \frac{1}{2} \varepsilon E^2 \right) \]
\[ \vec{E} \cdot \vec{J} = \vec{E} \cdot (\sigma \vec{E}) = \sigma E^2 \]

we have
\[ \nabla \cdot (\vec{E} \times \vec{H}) = -\frac{\partial}{\partial t} \left( \frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2 \right) - \sigma E^2 \]

or
\[ \oint_S (\vec{E} \times \vec{H}) \cdot d\vec{s} = -\frac{\partial}{\partial t} \int \left( \frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2 \right) dv - \int \sigma E^2 dv \]
\[ \int \left( \frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2 \right) dv : \text{stored energy} \]

\[ \int \sigma E^2 dv : \text{energy dissipated} \]

By conservation of energy, define Poynting vector \( \vec{P} = \vec{E} \times \vec{H} \), the power density vector associated with an electromagnetic field.

For time-harmonic wave

\[
\vec{P}(z,t) = \Re\left[ \vec{E}(z)e^{j\omega t} \times \Re[\vec{H}(z)e^{j\omega t}] \right]
\]

\[
= \frac{1}{2} [\vec{E}(z)e^{j\omega t} + \overline{\vec{E}(z)e^{j\omega t}}] \times \frac{1}{2} [\vec{H}(z)e^{j\omega t} + \overline{\vec{H}(z)e^{j\omega t}}]
\]

\[
= \frac{1}{4} [\vec{E}(z)e^{j\omega t} \times \overline{\vec{H}(z)e^{j\omega t}} + \overline{\vec{E}(z)e^{j\omega t}} \times \vec{H}(z)e^{j\omega t} + \vec{E}(z)e^{j\omega t} \times \overline{\vec{H}(z)e^{j\omega t}} + \overline{\vec{E}(z)e^{j\omega t}} \times \vec{H}(z)e^{j\omega t}]
\]

\[
= \frac{1}{2} \Re[\vec{E}(z) \times \overline{\vec{H}^*(z)} + \overline{\vec{E}(z)} \times \vec{H}(z)e^{2j\omega t}]
\]

thus,

\[
\vec{P}_{av}(z,t) = \frac{1}{T} \int_0^T \vec{P}(z,t) \, dt = \frac{1}{2} \Re[\vec{E}(z) \times \overline{\vec{H}^*(z)}]
\]

the time average power density.

Like wise

\[
\oint_S (\vec{E} \times \overline{\vec{H}^*}) \cdot d\vec{s} = 2j\omega \int_V \left( \frac{1}{2} \varepsilon |E|^2 - \frac{1}{2} \mu |H|^2 \right) dv - \int_V \sigma E^2 dv
\]

Boundary Conditions
\[ \hat{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = -\mathbf{M}_s \]
\[ \hat{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s \]
\[ \hat{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \rho_s \]
\[ \hat{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = \rho_{ms} \]
Reciprocity Theorem

Two sets of solutions with two sets of excitation:

\[ \vec{E}_1, \vec{H}_1, \vec{J}_1, \vec{M}_1 \]
\[ \vec{E}_2, \vec{H}_2, \vec{J}_2, \vec{M}_2 \]

satisfying Maxwell’s Equations

\[ \nabla \times \vec{E}_1 = -j \omega \mu \vec{H}_1 - \vec{M}_1 \]
\[ \nabla \times \vec{H}_1 = j \omega \epsilon \vec{E}_1 + \vec{J}_1 \]
\[ \nabla \times \vec{E}_2 = -j \omega \mu \vec{H}_2 - \vec{M}_2 \]
\[ \nabla \times \vec{H}_2 = j \omega \epsilon \vec{E}_2 + \vec{J}_2 \]

Then

\[ \int_{\nu} \nabla \cdot (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1) \, dv = \oint_{S} (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1) \cdot ds \]
\[ = \int_{\nu} (\vec{E}_2 \cdot \vec{J}_1 - \vec{E}_1 \cdot \vec{J}_2 + \vec{H}_1 \cdot \vec{M}_2 - \vec{H}_2 \cdot \vec{M}_1) \, dv \]

1. No sources
\[ \oint_{S} (\vec{E}_1 \times \vec{H}_2) \cdot ds = \oint_{S} (\vec{E}_2 \times \vec{H}_1) \cdot ds \]
2. Bound by a perfect conductor
\[ \int_{\nu} (\vec{E}_1 \cdot \vec{J}_2 - \vec{H}_1 \cdot \vec{M}_2) \, dv = \int_{\nu} (\vec{E}_2 \cdot \vec{J}_1 - \vec{H}_2 \cdot \vec{M}_1) \, dv \]
3. Unbounded
\[
\int_{\mathcal{V}} (\vec{E}_1 \cdot \vec{J}_2 - \vec{H}_1 \cdot \vec{M}_2) \, d\mathcal{V} = \int_{\mathcal{V}} (\vec{E}_2 \cdot \vec{J}_1 - \vec{H}_2 \cdot \vec{M}_1) \, d\mathcal{V}
\]

**Uniqueness Theorem**

Let \(\vec{E}_a, \vec{H}_a, \vec{E}_b, \vec{H}_b\) be two sets of solutions of the same excitation, then

\[
\oint_{\mathcal{S}} (\vec{E}_a - \vec{E}_b) \times (\vec{H}_a - \vec{H}_b) \cdot d\mathcal{S} + j\omega \int_{\mathcal{V}} (\mu |\vec{H}_a - \vec{H}_b|^2 - \varepsilon |\vec{E}_a - \vec{E}_b|^2) \, d\mathcal{V} = 0
\]

The surface integral vanishes if
1. \(\hat{n} \times \vec{E} = \vec{E}_t\), tangential electric field equals specified
2. \(\hat{n} \times \vec{H} = \vec{H}_t\), tangential magnetic field specified.

then

\(\vec{E}_a = \vec{E}_b, \vec{H}_a = \vec{H}_b\)

That is, in a enclosed volume, if the source in the volume and the tangential fields on the boundary are the same, the fields are the same everywhere inside the volume.
**Image Theory**: an application of Uniqueness Theorem.

![Diagram](image)

**FIGURE 1.17** Electric and magnetic current images. (a) An electric current parallel to a ground plane. (b) An electric current normal to a ground plane. (c) A magnetic current parallel to a ground plane. (d) A magnetic current normal to a ground plane.
Transmission Line Theory

\[ R \text{: series resistance per unit length in } \Omega/m. \]
\[ L \text{: series inductance per unit length in } \text{H/m}. \]
\[ G \text{: shunt conductance per unit length in } \text{S/m}. \]
\[ C \text{: shunt capacitance per unit length in } \text{F/m}. \]

By Kirchhoff’s voltage law:
\[ v(z,t) - R\Delta z i(z,t) - L\Delta z \frac{\partial i(z,t)}{\partial t} - v(z + \Delta z, t) = 0 \]

By Kirchhoff’s current law:
\[ i(z,t) - G\Delta z v(z + \Delta z,t) - C\Delta z \frac{\partial v(z + \Delta z,t)}{\partial t} - i(z + \Delta z,t) = 0 \]

As \( \Delta z \to 0 \),
\[ \frac{\partial v(z,t)}{\partial z} = -R i(z,t) - L \frac{\partial i(z,t)}{\partial t} \]
\[ \frac{\partial i(z,t)}{\partial z} = -G v(z,t) - C \frac{\partial v(z,t)}{\partial t} \]

For time-harmonic circuits
\[
\frac{dV(z)}{dz} = -(R + j\omega L)I(z)
\]
\[
\frac{dI(z)}{dz} = -(G + j\omega C)V(z)
\]

Thus
\[
\frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0
\]
\[
\frac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0
\]

where
\[
\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}: \text{complex propagation constant.}
\]

We have the solutions
\[
V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}
\]
\[
I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}
\]

\(e^{-\gamma z}\): positive z-direction propagation wave.
\(e^{\gamma z}\): negative z-direction propagation wave.

\(V_0^+, V_0^-, I_0^+, I_0^-\): constants.

Also
\[
I(z) = \frac{\gamma}{R + j\omega L} \left[ V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z} \right]
\]

Define characteristic impedance
\[
Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}
\]

Then
\[
\frac{V_0^+}{I_0^+} = \frac{V_0^-}{I_0^-} = Z_0 \quad \text{and} \quad I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}
\]

For lossless line
\(R = G = 0\)
\[ \alpha = 0, \quad \beta = \omega \sqrt{LC}, \quad Z_0 = \sqrt{\frac{L}{C}} \]

**Terminated Lossless Transmission Line**

![Diagram of a terminated lossless transmission line with notation for impedance and phase shift]

Assume incident wave \( V_0^+ e^{-j\beta z} \), reflected wave \( V_0^- e^{j\beta z} \) then

\[
V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}
\]

\[
I(z) = \frac{V_0^+ e^{-j\beta z} - V_0^- e^{j\beta z}}{Z_0}
\]

At \( z = 0 \),

\[
\frac{V(0)}{I(0)} = \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} Z_0 = Z_L \rightarrow \Gamma = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0}
\]

Define return loss: \( RL = -20 \log |\Gamma| \) dB

\[
Z_{in}(-l) = \frac{V(-l)}{I(-l)} = \frac{V_0^+ (e^{j\beta l} + \Gamma e^{-j\beta l})}{V_0^+ (e^{j\beta l} - \Gamma e^{-j\beta l})} Z_0 = Z_0 Z_L + jZ_0 \tan \beta l
\]

Special case:
1. \( Z_L = 0 \) (short): \( Z_{in} = jZ_0 \tan \beta l \).
2. \( Z_L = \infty \) (open): \( Z_{in} = -jZ_0 \cot \beta l \).
3. Half wavelength line: \( Z_{in} = Z_L \).
4. Quarter wavelength line: \( Z_m = \frac{Z_0^2}{Z_L} \)

Two-transmission Line Junction

\[ V(z) = V_0^+ (e^{-j\beta z} + \Gamma e^{j\beta z}), \text{ for } z < 0 \]

\[ V(z) = V_0^+ Te^{-j\beta z}, \text{ for } z > 0 \]

\[ \Gamma = \frac{Z_1 - Z_0}{Z_0 + Z_1} \]

At \( z = 0 \),

\[ T = 1 + \Gamma = \frac{2Z_1}{Z_1 + Z_0} \quad \text{transmission coefficient}. \]

Define Insertion loss: \( IL = -20\log|T| \) dB

Conservation of energy

Incident power: \( \frac{1}{2} \frac{|V_0^+|^2}{Z_0} \)

Reflected power: \( \frac{1}{2} \frac{|V_0^+\Gamma|^2}{Z_0} \)

Transmitted power: \( \frac{1}{2} \frac{|V_0^+T|^2}{Z_1} \)
Voltage Standing Wave Ratio (VSWR)

$$|V(z)| = |V_0^*||1 + e^{2j\beta z}|$$

$$V_{\text{max}} = |V_0^*|(1 + |\Gamma|), \quad V_{\text{in}} = |V_0^*|(1 - |\Gamma|)$$

Define Standing Wave Ratio

$$\text{SWR} = \frac{V_{\text{max}}}{V_{\text{min}}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

Smith Chart

Suppose a transmission line terminated by a load impedance $Z_L$.

Define normalized impedance $z_L = \frac{Z_L}{Z_0} = r_L + jx_L$, where $Z_0$ is the characteristic impedance of the transmission line. Then,

$$\Gamma = \frac{z_L - 1}{z_L + 1} = |\Gamma| e^{j\theta} = \Gamma_r + j\Gamma_i \Rightarrow r_L + jx_L = \frac{(1 + \Gamma_r) + j\Gamma_i}{(1 - \Gamma_r) - j\Gamma_i}$$

Equating the real and imaginary parts, we have

$$r_L = \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1 - \Gamma_r)^2 + \Gamma_i^2}$$

$$x_L = \frac{2\Gamma_i}{(1 - \Gamma_r)^2 + \Gamma_i^2}$$
Rearranging, we have

\[
\left( \Gamma_r - \frac{r_L}{1+r_L} \right)^2 + \Gamma_i^2 = \left( \frac{1}{1+r_L} \right)^2
\]

\[
(\Gamma_r - 1)^2 + \left( \Gamma_i - \frac{1}{x_L} \right)^2 = \frac{1}{x_L^2}
\]

Summary

**Constant resistance circle**

1. Center: \( \left( \frac{r_L}{1+r_L}, 0 \right) \),
2. Radius: \( \frac{1}{1+r_L} \),
3. Always passes \((1,0)\),
4. \( r_L \) decreases, radius increase,
5. \( r_L = 0 \) (short), unit circle,
6. \( r_L = \infty \) (open), point \((1,0)\).

**Constant reactance circle**

1. Center: \( \left( 1, \frac{1}{x_L} \right) \),
2. Radius: \( \frac{1}{x_L} \),
3. Always passes: \((1,0)\),
4. \( x_L \) decreases, radius increase,
5. \( x_L = 0 \) (short), \( x \) axis,
6. \( x_L = \infty \) (open), point \((1,0)\).

Since \( Z_{in} = \frac{Z_0}{1 + \Gamma e^{-2j\beta l}} \Rightarrow \) a rotation of angle \( 2\beta l \)
clockwise.

Calculation of VSWR:

\[ VSWR = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+\Gamma'}{1-\Gamma'} \rightarrow r_L' \text{, where } \Gamma' = |\Gamma|. \text{ Therefore, } VSWR \text{ is the resistance value at the intersection point of the positive } x \text{ and constant } |\Gamma| \text{ circle.} \]

Admittance Smith Chart

\[ y_L = \frac{1}{z_L} = \frac{1-\Gamma}{1+\Gamma} = \frac{1+\Gamma'}{1-\Gamma'} \text{, where } \Gamma' = \Gamma e^{j\pi}. \text{ Therefore, Admittance Smith Chart is a rotation of 180 degree of impedance Smith Chart.} \]
Example 2.2 Basic Smith Chart Operations
Example 2.3 Smith Chart Operations Using Admittances
Example 2.4 Impedance Measurement with a Slotted Line
FIGURE 2.13  An X-band waveguide slotted line.
Example 2.5 Frequency Response of a Quarter-wave Transformer
Generator and Load Mismatches

\[ P = \frac{1}{2} \text{Re} \left( V_{in} I_{in}^* \right) = \frac{1}{2} |V_{in}|^2 \text{Re} \left( \frac{1}{Z_{in}} \right) = \frac{1}{2} |V_{g}|^2 \frac{Z_{in}}{Z_{in} + Z_{g}} = \frac{1}{2} |V_{g}|^2 \frac{Z_{in}}{(R_{in} + R_{g})^2 + (X_{in} + X_{g})^2} \]

1. Load Matched to Line

\[ P = \frac{1}{2} |V_{g}|^2 \frac{Z_{0}}{(Z_{0} + R_{g})^2 + X_{g}^2} \]

2. Generator Matched to Loaded Line

\[ P = \frac{1}{2} |V_{g}|^2 \frac{R_{g}}{4(R_{g}^2 + X_{g}^2)} \]

3. Conjugate Matched

\[ \frac{\partial P}{\partial R_{in}} = 0 \Rightarrow R_{g}^2 - R_{in}^2 + (X_{in} + X_{g})^2 = 0 \]

\[ \frac{\partial P}{\partial X_{in}} = 0 \Rightarrow X_{in}(X_{in} + X_{g})^2 = 0 \]

\[ \Rightarrow R_{in} = R_{g}, X_{in} = -X_{g} \Rightarrow P = \frac{1}{2} |V_{g}|^2 \frac{1}{4R_{g}} \]

Note this result means maximum power delivered to the load under fixed \( V_{g} \). In reality, our concern is how much portion of total power is delivered to the load which is related to \( \frac{Z_{in}}{Z_{in} + Z_{g}} \).
Lossy Transmission Lines

Low-Loss Line

\[ \gamma \approx j \omega \sqrt{LC} \left[ 1 - \frac{j}{2} \left( \frac{R}{\omega L} + \frac{G}{\omega C} \right) \right] \]

\[ \alpha \approx \frac{1}{2} \left( R \sqrt{\frac{C}{L}} + G \sqrt{\frac{L}{C}} \right) \]

\[ \beta = \omega \sqrt{LC} \]

\[ Z_0 = \sqrt{\frac{L}{C}} \]

Distortionless Line

\[ \frac{R}{L} = \frac{G}{C} \]

\[ \gamma = R_s \sqrt{\frac{C}{L}} + j \omega \sqrt{LC} = \alpha + j \beta, \quad Z_0 = \sqrt{\frac{L}{C}}. \]

Method of Evaluation Attenuation Constants

1. Perturbation

\[ P(z) = P_0 e^{-2\alpha z} \]

Power loss per unit length:

\[ P_l = \frac{-\partial P}{\partial z} = 2\alpha P_0 e^{-2\alpha z} = 2\alpha P(z) \quad \Rightarrow \quad \alpha = \frac{P_l(z)}{2P(z)} = \frac{P_l(z=0)}{2P_0} \]

ex. 2.7

2. Wheeler Incremental Inductance Rule

\[ \alpha_c = \frac{\omega \Delta L}{2Z_0} \quad \text{or} \quad \alpha_c = \frac{R_s}{2Z_0} \int dZ_0, \quad \text{where} \ \Delta L \ \text{and} \ dZ_0 \ \text{are} \]

changes due to recess of all conductor walls by an
amount of $\frac{\delta_s}{2}$.

Ex. 2.8
Plane Waves in Lossy Media

If the material is conductive
\[ \nabla \times \vec{H} = j \omega \varepsilon \vec{E} + j \sigma \varepsilon \vec{E} = j \omega \varepsilon (1 + \frac{\sigma}{j \omega \varepsilon}) \vec{E} = j \omega \varepsilon \frac{\sigma}{j \omega \varepsilon} \vec{E}, \]
where \( \varepsilon_c = \varepsilon (1 + \frac{\sigma}{j \omega \varepsilon}) \) is the complex permittivity.

we have
\[ \gamma = \alpha + j \beta = j \omega \sqrt{\mu \varepsilon} \left( 1 + \frac{\sigma}{j \omega \varepsilon} \right)^{\frac{1}{2}} \]

Or if the material has dielectric loss with \( \varepsilon = \varepsilon' - j \varepsilon'' = \varepsilon' (1 - j \tan \delta) \)
\[ \gamma = \alpha + j \beta = j \omega \sqrt{\mu \varepsilon} \left( 1 - j \frac{\varepsilon''}{\varepsilon'} \right)^{\frac{1}{2}} \]
where \( \alpha \) is the attenuation constant, \( \beta \) the phase constant.
\( \tan \delta = \frac{\varepsilon''}{\varepsilon'} \) is the loss tangent.

Low-Loss Dielectrics: \( \varepsilon'' \ll \varepsilon' \), or \( \frac{\sigma}{\omega \varepsilon} \ll 1 \)
\[ \gamma = \alpha + j \beta = j \omega \sqrt{\mu \varepsilon} \left[ 1 - j \frac{\varepsilon''}{2 \varepsilon'} + \frac{1}{8} \left( \frac{\varepsilon''}{\varepsilon'} \right)^2 \right] \]
\[ \therefore \alpha = \frac{\omega \varepsilon''}{2} \sqrt{\frac{\mu}{\varepsilon'}} \beta = \omega \sqrt{\mu \varepsilon} \left[ 1 + \frac{1}{8} \left( \frac{\varepsilon''}{\varepsilon'} \right)^2 \right] \]

and
\[ \eta = \sqrt{\frac{\mu}{\varepsilon} \left( 1 - j \frac{\varepsilon''}{\varepsilon'} \right)} = \sqrt{\mu} \left( 1 + j \frac{\varepsilon''}{2 \varepsilon'} \right) \]
\[ u_p = \frac{\omega}{\beta} \sqrt{\mu \varepsilon} \left[ 1 - \frac{1}{8} \left( \frac{\varepsilon''}{\varepsilon'} \right)^2 \right] \]
Good Conductor: \( \frac{\sigma}{\omega \varepsilon} \to 1 \)

\[
\gamma = \alpha + j\beta = j\omega \sqrt{\mu \varepsilon} \sqrt{\frac{\sigma}{j\omega \varepsilon}} = \frac{1 + j}{\sqrt{2}} \sqrt{\mu \sigma}
\]

\[
\therefore \quad \alpha = \beta = \frac{\sqrt{\mu \sigma}}{2}
\]

and

\[
\eta = (1 + j) \frac{\alpha}{\sigma}
\]

\[
u_p = \sqrt{\frac{2\omega}{\mu \sigma}}
\]

Skin depth or depth of penetration:

\[
\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega \mu \sigma}}
\]

Meaning: plane wave decay be a factor of \( e^{-1} = 0.368 \). At microwave frequencies, \( \delta \) is very small for a good conductor, thus confined in a very thin layer of the conductor surface.

Let \( \sigma_s \) be the equivalent surface conductivity defined by

\[
J_s = \sigma_s E_0 = \int_{0}^{\infty} \sigma E_0 e^{-\alpha z} dz = \frac{\sigma}{\alpha} E_0 = \sqrt{\frac{2\sigma}{\omega \mu} E_0}
\]

Surface Resistance: \( R_s = \frac{1}{\sigma_s} = \sqrt{\frac{\omega \mu}{2\sigma}} \)
2.16 The transmission line circuit in the accompanying figure has $V_g = 15$ V rms, $Z_g = 75\ \Omega$, $Z_0 = 75\ \Omega$, $Z_L = 60 - j40\ \Omega$, and $\ell = 0.7\lambda$. Compute the power delivered to the load using three different techniques:

(a) Find $\Gamma$ and compute

$$P_L = \left(\frac{V_g}{2}\right)^2 \frac{1}{Z_0} (1 - |\Gamma|^2);$$

(b) find $Z_{in}$ and compute

$$P_L = \left|\frac{V_g}{Z_g + Z_{in}}\right|^2 \text{Re}\{Z_{in}\};$$

(c) find $V_L$ and compute

$$P_L = \left|\frac{V_L}{Z_L}\right|^2 \text{Re}\{Z_L\}.$$  

Discuss the rationale for each of these methods. Which of these methods can be used if the line is not lossless?

---

3.6 An attenuator can be made using a section of waveguide operating below cutoff, as shown in the accompanying figure. If $a = 2.286$ cm and the operating frequency is $12$ GHz, determine the required length of the below-cutoff section of waveguide to achieve an attenuation of $100$ dB between the input and output guides. Ignore the effect of reflections at the step discontinuities.
2.27 In Example 2.7, the attenuation of a coaxial line due to finite conductivity is

\[ \alpha_c = \frac{R_s}{2\eta \ln \frac{b}{a}} \left( \frac{1}{a} + \frac{1}{b} \right) . \]

Show that \( \alpha_c \) is minimized for conductor radii such that \( x \ln x = 1 + x \), where \( x = b/a \). Solve this equation for \( x \), and show that the corresponding characteristic impedance for \( \epsilon_r = 1 \) is 77 \( \Omega \).

3.28 As discussed in the Point of Interest on the power-handling capacity of transmission lines, the maximum power capacity of a coaxial line is limited by voltage breakdown and is given by

\[ P_{\text{max}} = \frac{\pi a^2 E_d^2}{\eta_0 \ln \frac{b}{a}} , \]

where \( E_d \) is the field strength at breakdown. Find the value of \( b/a \) that maximizes the maximum power capacity and show that the corresponding characteristic impedance is about 30 \( \Omega \).
Transmission Lines and Waveguides

General Solutions for TEM, TE, and TM Waves

Rectangular Waveguides

Coaxial Lines

Microstrip

Strip Lines

Coplanar Waveguides
General Solutions for TEM, TE, and TM Waves

Assuming a wave propagating in the \( +z \) direction, the electric and magnetic fields can be expressed as

\[
\vec{E}(x, y, z) = [\vec{e}(x, y) + \hat{z} e_z(x, y)] e^{-j\beta z}
\]

\[
\vec{H}(x, y, z) = [\vec{h}(x, y) + \hat{z} h_z(x, y)] e^{-j\beta z}
\]

where \( \vec{e}(x, y) \) and \( \vec{h}(x, y) \) are the transverse \((\hat{x}, \hat{y})\) electric and magnetic field components.

In a source-free region, Maxwell’s equations can be written as

\[
\nabla \times \vec{E}(x, y, z) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -j \omega \mu \vec{H}(x, y, z)
\]

\[
\nabla \times \vec{H}(x, y, z) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = j \omega \varepsilon \vec{E}(x, y, z)
\]
With an $e^{-j\beta z}$ dependence in $z$ direction, the above equations can be reduced to the following:

\[
\begin{align*}
\frac{\partial E_z}{\partial y} + j\beta E_y &= -j\omega \mu H_x \\
-\frac{\partial E_z}{\partial x} - j\beta E_x &= -j\omega \mu H_y \\
\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -j\omega \mu H_z \\
-\frac{\partial H_z}{\partial y} + j\beta H_y &= j\omega \mu E_x \\
-\frac{\partial H_z}{\partial x} - j\beta H_x &= j\omega \mu E_y \\
\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= j\omega \mu E_z
\end{align*}
\]

Solving the four transverse field components in terms of $E_z$ and $H_z$, we have

\[
\begin{align*}
H_x &= \frac{j}{k_c^2} \left( \omega \varepsilon \frac{\partial E_z}{\partial y} - \beta \frac{\partial H_z}{\partial x} \right) \\
H_y &= -\frac{j}{k_c^2} \left( \omega \varepsilon \frac{\partial E_z}{\partial x} + \beta \frac{\partial H_z}{\partial y} \right) \\
E_x &= -\frac{j}{k_c^2} \left( \omega \mu \frac{\partial H_z}{\partial y} + \beta \frac{\partial E_z}{\partial x} \right) \\
E_y &= \frac{j}{k_c^2} \left( \omega \mu \frac{\partial H_z}{\partial x} - \beta \frac{\partial E_z}{\partial y} \right)
\end{align*}
\]
where
\[ k_c^2 = k^2 - \beta^2 \]
\[ k^2 = \omega^2 \mu \varepsilon \]

**Case 1.** \( E_z = H_z = 0 \) (Transverse Electromagnetic Waves)
\[ k_c^2 = 0 \Rightarrow \beta = k \]

Property: No cutoff frequency.

\[ \nabla_t \times \vec{\varepsilon}(x,y) = -j \omega \mu h_z(x,y) \hat{z} = 0 \Rightarrow \vec{\varepsilon}(x,y) = -\nabla_t \Phi(x,y) \]

where \( \nabla_t = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} \).

Property: Voltage can be uniquely defined.

From Maxwell’s equations
\[ \nabla_t \vec{D} = \varepsilon \nabla_t \vec{\varepsilon} = -\varepsilon \nabla_t \nabla_t \Phi = 0 \Rightarrow \nabla_t^2 \Phi = 0 \]

Property: \( \Phi(x,y) \) satisfies Laplace’s equation.

To sum up, the transverse fields of an TEM wave have the same properties of an electrostatic field except that it is in two dimension.

Define wave impedance
\[ Z_{TEM} = \frac{E_x}{H_y} = \frac{\omega \mu}{\beta} = \sqrt{\frac{\mu}{\varepsilon}} = \eta = -\frac{E_y}{H_x} \]
Thus, \( \vec{h}(x,y) = \frac{1}{Z_{TEM}} \hat{z} \times \vec{e}(x,y) \)

Property: The phase constant, wave impedance and relationship of electric field and magnetic field are the same as an plane wave.

Property: TEM waves can exist when two or more conductors are present.

**Case 2.** \( E_z = 0, \quad H_z \neq 0 \) (Transverse Electric Waves)

\[
H_x = -j \frac{\beta}{k_c^2} \frac{\partial H_z}{\partial x} \\
H_y = -j \frac{\beta}{k_c^2} \frac{\partial H_z}{\partial y} \\
E_x = -j \frac{\omega \mu}{k_c^2} \frac{\partial H_z}{\partial y} \\
E_y = j \frac{\omega \mu}{k_c^2} \frac{\partial H_z}{\partial x}
\]

where

\[
k_c \neq 0 \Rightarrow \beta = \sqrt{k^2 - k_c^2}
\]

Property: \( k_c \) is a function of the physical structure of the waveguide and frequency. For a fixed \( k_c \),
when \( k^2 > k_c^2 \), \( \beta \) is real and when \( k^2 < k_c^2 \), \( \beta \) is imaginary, not a propagating wave. At \( k^2 = \omega_c^2 \mu \varepsilon = k_c^2 \), the wave stops to propagate, we call this \( \omega_c \) the cutoff frequency.

Solving \( H_z \) from the Helmholtz wave equation,

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) H_z = 0 \quad \Rightarrow \quad \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2 \right) h_z = 0
\]

This equation must satisfy the boundary conditions of the specific guide geometry.

Define TE wave impedance

\[
Z_{TE} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\omega \mu}{\beta} = \frac{k \eta}{\beta}
\]

**Case 3.** \( H_z = 0, \ E_z \neq 0 \) (Transverse Magnetic Waves)
Solving $E_z$ from the Helmholtz wave equation,

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k_c^2 \right) E_z = 0 \Rightarrow \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2 \right) e_z = 0
\]

This equation must satisfy the boundary conditions of the specific guide geometry.

Define TM wave impedance

\[
Z_{TM} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\beta}{\omega \varepsilon} = \frac{\beta \eta}{k}
\]

**Rectangular Waveguides**
Task: solve \( \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2 \right) h_z = 0 \)

Assume
\[ h_z(x, y) = X(x)Y(y) \]

Substitute to the above equation, we have
\[ \frac{1}{X} \frac{d^2X}{dx^2} + \frac{1}{Y} \frac{d^2Y}{dy^2} + k_c^2 = 0 \]

Possible solution of the above equation is
\[ \frac{d^2X}{dx^2}+k_x^2X=0 \quad \Rightarrow \quad X(x) = A \cos k_x x + B \sin k_x x \]
\[ \frac{d^2Y}{dy^2}+k_y^2Y=0 \quad \Rightarrow \quad Y(y) = C \cos k_y y + D \sin k_y y \]

Thus
\[ k_x^2 + k_y^2 = k_c^2 \]
\[ h_z(x, y) = (A \cos k_x x + B \sin k_x x)(C \cos k_y y + D \sin k_y y) \]

From boundary conditions

\[
\begin{align*}
    e_x(x, y) &= 0, \text{ at } y = 0, b \\
    e_y(x, y) &= 0, \text{ at } x = 0, a
\end{align*}
\]

\[
\begin{align*}
    B &= D = 0 \\
    k_x &= \frac{m \pi}{a} \\
    k_y &= \frac{n \pi}{b}
\end{align*}
\]

That is

\[ H_z(x, y, z) = A_{mn} \cos \frac{m \pi x}{a} \cos \frac{n \pi y}{b} e^{-\beta z} \]

where \( A_{mn} \) is an arbitrary constant, \( m = 0, 1, 2, \ldots \) and \( n = 0, 1, 2, \ldots \) except \( m = n = 0 \)

The propagating constant is

\[
\beta = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m \pi}{a}\right)^2 - \left(\frac{n \pi}{b}\right)^2} = k \sqrt{1 - \left(\frac{f_{cmn}}{f}\right)^2}
\]

Cut-off frequency

\[
f_{cmn} = \frac{k_c}{2 \pi \sqrt{\mu \varepsilon}} = \frac{1}{2 \pi \sqrt{\mu \varepsilon}} \sqrt{\left(\frac{m \pi}{a}\right)^2 + \left(\frac{n \pi}{b}\right)^2}
\]

Guide wavelength

\[
\lambda_g = \frac{2 \pi}{\beta} = \frac{\lambda}{\sqrt{1 - \left(\frac{f_{cmn}}{f}\right)^2}} > \lambda
\]

Phase velocity
The mode with the lowest cutoff frequency is called the dominant mode or the fundamental mode, which is the $TE_{10}$ ($m=1, n=0$) mode.

\[
f_{c_{10}} = \frac{1}{2a\sqrt{\mu\varepsilon}}
\]

Group velocity

\[
\nu_g = \frac{1}{\sqrt{\mu\varepsilon}} \left( \frac{f_{c_{mn}}}{f} \right)^2 < \frac{1}{\sqrt{\mu\varepsilon}}
\]
TM Modes

Task: solve \( \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2 \right) e_z = 0 \)

Assume 
\( h_z(x,y) = X(x)Y(y) \)

Substitute to the above equation, we have
\[
\frac{1}{X} \frac{d^2X}{dx^2} + \frac{1}{Y} \frac{d^2Y}{dy^2} + k_c^2 = 0
\]

Possible solution of the above equation is
\[
\frac{d^2X}{dx^2} + k_x^2 X = 0 \Rightarrow X(x) = A \cos k_x x + B \sin k_x x
\]
\[
\frac{d^2Y}{dy^2} + k_y^2 Y = 0 \Rightarrow Y(y) = C \cos k_y y + D \sin k_y y
\]

Thus
\[
e_z(x,y) = (A \cos k_x x + B \sin k_x x)(C \cos k_y y + D \sin k_y y)
\]

From boundary conditions
\[
\begin{cases}
e_z(x,y) = 0, \text{ at } y = 0, b \\
e_z(x,y) = 0, \text{ at } x = 0, a
\end{cases} \Rightarrow
\begin{cases}
A = C = 0 \\
k_x = \frac{m \pi}{a} \\
k_y = \frac{m \pi}{b}
\end{cases}
\]

That is
where $A_{mn}$ is an arbitrary constant, $m=1,2,...$ and $n=1,2,...$.

The propagating constant, cutoff frequency, guide wavelength, phase velocity and group velocity are the same as TE modes.

The mode with the lowest cutoff frequency is the $TM_{11}$ ($m=1,n=1$) mode.

$$f_{c_{11}} = \frac{1}{2\pi\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2}$$
Loss in a Waveguide

Dielectric Loss

Let $\tan\delta \ll 1$ be the loss tangent of a dielectric. The complex propagation constant $\gamma$ can be expressed as

$$\gamma = \alpha_d + j\beta = \sqrt{k_c^2 - k_0^2} = \sqrt{k_c^2 - \omega^2 \mu_0 \varepsilon_0 \varepsilon_r (1 - j\tan\delta)}$$

Since $\tan\delta \ll 1$ we have

$$\gamma = \frac{\omega^2 \mu_0 \varepsilon_0 \varepsilon_r \tan\delta}{2\beta} + j\beta$$

where $j\beta = \sqrt{k_c^2 - \omega^2 \mu_0 \varepsilon_0 \varepsilon_r}$. Thus

$$\alpha_d = \frac{\omega^2 \mu_0 \varepsilon_0 \varepsilon_r \tan\delta}{2\beta}$$

is the attenuation constant due to dielectric loss.

Conductor Loss

Let power flow be

$$P(z) = P_0 e^{-2\alpha_c z}$$

Then the power loss per unit length along the line is

$$P_\ell(z) = -\frac{\partial P}{\partial z} = 2\alpha_c P_0 e^{-2\alpha_c z} = 2\alpha_c P(z) \to \alpha_c = \frac{P_\ell(z)}{2P(z)} = \frac{P_\ell(z=0)}{2P_0}$$

The power lost in the conductor due to the surface
resistance \( R_s = \sqrt{\frac{\omega \mu_0}{2\sigma}} \) (\( \sigma \) is the conductance of the conductor).

\[
P_s = \frac{R_s}{2} \int_S |\vec{J}|^2 \, ds
\]

**Total Loss** \( a = a_c + a_d \)

**\( \text{TE}_{10} \) modes**

\[
H_z = A_{10} \cos \frac{\pi x}{a} e^{-j\beta z}
\]

\[
E_y = \frac{-j \omega \mu a}{\pi} A_{10} \sin \frac{\pi x}{a} e^{-j\beta z}
\]

\[
H_x = \frac{j \beta a}{\pi} A_{10} \sin \frac{\pi x}{a} e^{-j\beta z}
\]

\( E_x = E_z = H_y = 0 \)
Coaxial Line

TEM mode

Let $a$ be the inner radius of the coaxial line and $b$ be the outer radius of the coaxial line. Let $\Phi$ be the potential function of the TEM mode, then $\Phi$ satisfies Laplace’s equation $\nabla^2 \Phi = 0$. In polar coordinate

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \Phi(\rho, \varphi)}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi(\rho, \varphi)}{\partial \varphi^2} = 0$$

and the boundary condition

$\Phi(a, \varphi) = V_0$

$\Phi(b, \varphi) = 0$

Due to symmetry, $\Phi(\rho, \varphi) = R(\rho)$, we have

$$\frac{\rho}{R} \frac{\partial}{\partial \rho} \left( \rho \frac{dR}{d\rho} \right) = 0 \Rightarrow R(\rho) = A \ln \rho + B$$

Use the boundary condition to solve $A$ and $B$, we have

$$\Phi(\rho, \varphi) = \frac{V_0 \ln \frac{b}{a}}{\ln \frac{b}{a}} \rho$$
\( \vec{v}(\rho, \phi) = -\nabla_t \Phi(\rho, \phi) = \frac{V_0 \hat{\rho}}{\rho \ln \frac{b}{a}} \)

\( \vec{h}(\rho, \phi) = \frac{1}{\eta} \times \vec{v}(\rho, \phi) = \frac{V_0 \hat{\phi}}{\eta \rho \ln \frac{b}{a}} \)

\[ I_a = \int_{\phi=0}^{2\pi} H_\phi(a, \phi) a d\phi = \frac{2\pi V_0}{\eta \ln \frac{b}{a}} = -I_b \]

\[ Z_0 = \frac{V_0}{I_a} = \frac{\eta \ln \frac{b}{a}}{2\pi} \]

\[ C = \varepsilon \frac{2\pi}{\ln \frac{b}{a}} \]

\[ L = \mu \frac{\ln \frac{b}{a}}{2\pi} \]
Microstrip Line

Formulas

\[ v_p = \frac{c}{\sqrt{\varepsilon_e}}, \quad \beta = k_0 \sqrt{\varepsilon_e} \]

\[ \varepsilon_e = \frac{\varepsilon_r + 1}{2} \cdot \frac{\varepsilon_r - 1}{2} \cdot \frac{1}{\sqrt{1 + 12d/W}} \]

\[ Z_0 = \sqrt{\frac{120\pi}{\varepsilon_e \left[ \frac{W}{d} + 1.393 + 0.667\ln\left( \frac{W}{d} + 1.444 \right) \right]}} \quad \text{for} \quad \frac{W}{d} \geq 1 \]

Or

\[ \frac{W}{d} = \frac{8e^A}{e^{2A} - 2} \quad \text{for} \quad \frac{W}{d} < 2 \]

\[ \frac{2}{\pi} \left[ B - 1 - \ln(2B - 1) + \frac{\varepsilon_r - 1}{2\varepsilon_r} \left\{ \ln(B - 1) + 0.39 - 0.61 \frac{0.39}{\varepsilon_r} \right\} \right] \quad \text{for} \quad \frac{W}{d} > 2 \]

where

\[ A = \frac{Z_0}{60} \left[ \frac{\varepsilon_r + 1}{2} + \frac{\varepsilon_r - 1}{\varepsilon_r + 1} \left( 0.23 + \frac{0.11}{\varepsilon_r} \right) \right] \]
\[ B = \frac{377\pi}{2Z_0\sqrt{\varepsilon_r}} \]

Loss

\[ \alpha_d = \frac{k_0\varepsilon_r(\varepsilon_r - 1)\tan\delta}{2\sqrt{\varepsilon_r(\varepsilon_r - 1)}} \text{ Np/m} \]
\[ \alpha_c = \frac{R_s}{Z_0W} \text{ Np/m} \]

where

\[ R_s = \sqrt{\frac{\omega\mu_0}{2\sigma}} \]

Operating frequency limits

The lower-order strong coupled TM mode:

\[ f = \frac{c\tan^{-1}\varepsilon_r}{\sqrt{2\pi d\sqrt{\varepsilon_r - 1}}} \]

The lowest-order transverse microstrip resonance:

\[ f = \frac{c}{\sqrt{\varepsilon_r(2W + 0.8d)}} \]

Frequency Dependence

\[ \varepsilon_{eff}(f) = \left( \frac{\sqrt{\varepsilon_r - \sqrt{\varepsilon_{eff}(0)}}}{1 + 4F^{-1.5}} + \sqrt{\varepsilon_{eff}(0)} \right)^2 \]

\[ Z_0(f) = Z_0 \frac{\varepsilon_{eff}(f) - 1}{\varepsilon_{eff}(0) - 1} \left( \frac{\varepsilon_{eff}(0)}{\varepsilon_{eff}(f)} \right) \]

where

\[ F = \frac{4d\sqrt{\varepsilon_r - 1}}{\lambda_0} \left\{ 0.5 + \left[ 1 + 2 \log_{10} \left( 1 + \frac{W}{d} \right) \right] \right\} \]
Strip Line

Formulas

\[ v_p = \frac{c}{\sqrt{\varepsilon_r}} \]

\[ \beta = \sqrt{\varepsilon_r} k_0 \]

\[ Z_0 = \sqrt{\frac{L}{C}} = \frac{\sqrt{LC}}{C} = \frac{1}{\sqrt{\varepsilon_r} v_p} C = \frac{30\pi}{\sqrt{\varepsilon_r} W_s + 0.441b} \]

where

\[ \frac{W_e}{b} = \begin{cases} 
0 & \text{for } \frac{W}{b} > 0.35 \\
 (0.35 - \frac{W}{b})^2 & \text{for } \frac{W}{b} < 0.35 
\end{cases} \]

Or

\[ \frac{W}{b} = \begin{cases} 
x & \text{for } \sqrt{\varepsilon_r} Z_0 < 120 \\
0.85 - \sqrt{0.6 - x} & \text{for } \sqrt{\varepsilon_r} Z_0 > 120 
\end{cases} \]

where

\[ x = \frac{30\pi}{\sqrt{\varepsilon_r} Z_0} - 0.441 \]

Loss

\[ \alpha_c = \begin{cases} 
2.7 \times 10^{-3} R_s \varepsilon_r Z_0 A & \text{for } \sqrt{\varepsilon_r} Z_0 < 120 \\
\frac{0.16 R_s B}{Z_0 b} & \text{for } \sqrt{\varepsilon_r} Z_0 > 120 
\end{cases} \text{ Np/m} \]
where

\[
A = 1 + \frac{2W}{b-t} + \frac{1}{\pi} \frac{b+t}{b-t} \ln \left( \frac{2b-t}{t} \right)
\]

\[
B = 1 + \frac{b}{0.5W + 0.7t} \left( 0.5 + \frac{0.414t}{W} + \frac{1}{2\pi} \ln \frac{4\pi W}{t} \right)
\]

\[
\alpha_d = \frac{kt \tan \delta}{2} \text{ Np/m}
\]
**Coplanar Waveguide (CPW)**

**Benefit:**
1. Lower dispersion.
2. Convenient connecting lump circuit elements.

![Coplanar Waveguide geometry](image)

**TABLE 3.6 Comparison of Common Transmission Lines and Waveguides**

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>Coax</th>
<th>Waveguide</th>
<th>Stripline</th>
<th>Microstrip</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modes: Preferred</td>
<td>TEM</td>
<td>TE&lt;sub&gt;10&lt;/sub&gt;</td>
<td>TEM</td>
<td>Quasi-TEM</td>
</tr>
<tr>
<td>Other</td>
<td>TM,TE</td>
<td>TM,TE</td>
<td>TM,TE</td>
<td>Hybrid TM,TE</td>
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<td>Dispersion</td>
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<td>None</td>
<td>Low</td>
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<td>Bandwidth</td>
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<td>High</td>
<td>High</td>
</tr>
<tr>
<td>Loss</td>
<td>Medium</td>
<td>Low</td>
<td>High</td>
<td>High</td>
</tr>
<tr>
<td>Power capacity</td>
<td>Medium</td>
<td>High</td>
<td>Low</td>
<td>Low</td>
</tr>
<tr>
<td>Physical size</td>
<td>Large</td>
<td>Large</td>
<td>Medium</td>
<td>Small</td>
</tr>
<tr>
<td>Ease of fabrication</td>
<td>Medium</td>
<td>Medium</td>
<td>Easy</td>
<td>Easy</td>
</tr>
<tr>
<td>Integration with</td>
<td>Hard</td>
<td>Hard</td>
<td>Fair</td>
<td>Easy</td>
</tr>
<tr>
<td>other components</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Microwave Network Analysis

1. General Properties
2. Waveguide Discontinuity
3. Excitation of Waveguide
Impedance and Equivalent Voltages and Currents

Equivalent Voltages and Currents

\[ \vec{E}_t(x,y,z) = \vec{e}(x,y)(A + e^{-j\beta z} + A - e^{j\beta z}) = \frac{\vec{e}(x,y)}{C_1}(V^+ e^{-j\beta z} + V^- e^{j\beta z}) \]

\[ \vec{H}_t(x,y,z) = \vec{h}(x,y)(A + e^{-j\beta z} - A - e^{j\beta z}) = \frac{\vec{h}(x,y)}{C_2}(I^+ e^{-j\beta z} - I^- e^{j\beta z}) \]

\[ P^* = \frac{1}{2} \mathcal{A}^+ \int \int_{s} \vec{e} \times \vec{h}^* \cdot \hat{z} \, ds = \frac{V^{**} I^{**}}{2C_1 C_2^*} \int \int_{s} \vec{e} \times \vec{h}^* \cdot \hat{z} \, ds \]

Let \( P^* = \frac{1}{2} V^* I^{**} \), we have

\[ C_1 C_2^* = \int \int_{s} \vec{e} \times \vec{h}^* \cdot \hat{z} \, ds \]

Also

\[ Z_0 = \frac{V^*}{I^+} = -\frac{V^-}{I^-} = \frac{C_1}{C_2} \]

To solve \( C_1 \) and \( C_2 \), choose

\[ \frac{C_1}{C_2} = Z_w (Z_{TE} \text{ or } Z_{TM}), \text{ or} \]

\[ \frac{C_1}{C_2} = 1 \]

Example 4.1
Choose $Z_0 = Z_{TE} \Rightarrow \frac{C_1}{C_2} = Z_{TE} \Rightarrow$

\[
\begin{align*}
C_1 &= \sqrt{\frac{ab}{2}} \\
C_2 &= \frac{1}{Z_{TE}} \sqrt{\frac{ab}{2}}
\end{align*}
\]

**Concept of Impedance**

1. **Intrinsic impedance:** \( \eta = \sqrt{\frac{\mu}{\varepsilon}} \)

2. **Wave impedance:** \( Z_w = \frac{E_t}{H_t} \)

3. **Characteristic impedance:** \( Z_0 = \frac{V^*}{I^*} \)

Example 4.2
Properties of One Port

Complex power

\[ P = \frac{1}{2} \oint_S \vec{E} \times \vec{H}^* \cdot d\vec{s} = P_\ell + 2j\omega (W_m - W_e) \]

where

- \( P_\ell \): real positive. The average power dissipated.
- \( W_m \): real positive. The stored magnetic energy.
- \( W_e \): real positive. The stored electric energy.

Define real transverse model fields \( \vec{\epsilon} \) and \( \vec{h} \) such that

\[
\vec{E}_i(x,y,z) = V(z) \vec{\epsilon}(x,y)e^{-j\beta z} \\
\vec{H}_i(x,y,z) = I(z) \vec{\epsilon}(x,y)h^{-j\beta z}
\]

and

\[
\int_S \vec{\epsilon} \times \vec{h} \cdot d\vec{s} = 1
\]

then,

\[
P = \frac{1}{2} \int_S VI^* \vec{\epsilon} \times \vec{h} \cdot d\vec{s} = \frac{1}{2} VI^*
\]

Thus, the input impedance

\[
Z_{in} = R + jX = \frac{V}{I} = \frac{VI^*}{|I|^2} = \frac{P}{\frac{1}{2} |\eta|^2} = \frac{1}{2} \frac{|\eta|^2}{|\eta|^2} + 2j\omega (W_m - W_e)
\]

Properties:

1. \( R \) is related to \( P_\ell \). \( R \) equals zero if lossless.

2. \( X \) is related to \( 2j\omega (W_m - W_e) \). \( W_m > W_e \), inductive load. \( W_m < W_e \), capacitive load.

Even and Odd Properties of \( Z(\omega) \) and \( \Gamma(\omega) \)
\[ v(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} V(\omega) e^{j\omega t} d\omega \implies V(-\omega) = V^*(\omega) \text{ since } v(t) = v(t)^* \text{. Similarly,} \]

\[ I(-\omega) = I^*(\omega). \]

\[ V^*(\omega) = Z^*(\omega) I^*(\omega) = Z^*(\omega) I(-\omega) = Z(-\omega) I(-\omega) = V(-\omega) \]

\[ \therefore Z^*(\omega) = Z(-\omega) \]

\[ \Gamma(-\omega) = \frac{Z(-\omega) - Z_0(-\omega)}{Z(-\omega) + Z_0(-\omega)} = \Gamma^*(\omega) \]

Summary

1. Even functions: \( \Re(Z), \Re(V), \Re(I), \Re(\Gamma) \).
2. Odd functions: \( \Im(Z), \Im(V), \Im(I), \Im(\Gamma) \).
3. Even functions: \( |\Gamma(\omega)|, |V(\omega)|, |I(\omega)|, |Z(\omega)| \).

Properties of N-Port

Define impedance matrix \([Z]\)

\[
\begin{bmatrix}
V_1 \\
V_2 \\
\vdots \\
V_N
\end{bmatrix} = 
\begin{bmatrix}
Z_{11} & Z_{12} & \cdots & Z_{1N} \\
Z_{21} & Z_{22} & \cdots & Z_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
Z_{N1} & Z_{N2} & \cdots & Z_{NN}
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2 \\
\vdots \\
I_N
\end{bmatrix}
\]

where

\[
V_i = V_i^* + V_i^-, \quad I_i = I_i^* - I_i^-, \quad Z_{ij} = \left| \frac{V_i}{I_j} \right|_{I_k = 0 \text{ for } k \neq j}
\]

and admittance matrix \([Y]\)
Reciprocal Networks

Conditions:
1. No source in the network.
2. No ferrite or plasma.

\[ Z_{ij} = Z_{ji} \]
\[ Y_{ij} = Y_{ji} \]

Lossless networks: \( \Re(Z_{ij}) = 0 \), \( \Re(Y_{ij}) = 0 \)

Example 4.3
The Scattering Matrix

Define impedance matrix $[S]$

$$
\begin{bmatrix}
V_1^- \\
V_2^- \\
\vdots \\
V_N^-
\end{bmatrix} =
\begin{bmatrix}
S_{11} & S_{12} & \cdots & S_{1N} \\
S_{21} & S_{22} & \cdots & S_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
S_{N1} & S_{N2} & \cdots & S_{NN}
\end{bmatrix}
\begin{bmatrix}
V_1^+ \\
V_2^+ \\
\vdots \\
V_N^+
\end{bmatrix}
$$

where

$$S_{ij} = \frac{V_i^-}{V_j^+} \quad V_k^+ = 0 \text{ for } k \neq j$$

Relationship with $[Z]$

$$
\begin{bmatrix}
Z_{01} & 0 & 0 & \cdots & 0 \\
0 & Z_{02} & 0 & \cdots & 0 \\
0 & 0 & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & Z_{0N}
\end{bmatrix}
$$

Let $[Z_0]$ be the matrix formed by the characteristic impedance of each port.

$$
[V^+] + [V^-] = [V] = [Z][I] = [Z][(I^+)-(I^-)] = [Z][Z_0]^{-1}(V^+-V^-)
$$

Thus

$$
[V^-] = ([Z][Z_0]^{-1} + [U])^{-1}([Z][Z_0]^{-1} - [U]) [V^+]
$$

$$
[S] = ([Z][Z_0]^{-1} + [U])^{-1}([Z][Z_0]^{-1} - [U])
$$

Like wise,
\[
\left( [Z][Z_0]^{-1} + [U] \right)[S] = ( [Z][Z_0]^{-1} - [U] ) \\
\Rightarrow [U] + [S] = [Z][Z_0]^{-1}([U] - [S]) \\
\Rightarrow ( [U] + [S] ) ([U] - [S])^{-1} [Z_0] = [Z]
\]

If lossless
\[
P_{AV} = \frac{1}{2} \Re([V][V]^*) \\
= \frac{1}{2} \Re( [V^*]^T [V^*] [Z_0]^{-1} [V^*]^T [V^*] ) \\
= \frac{1}{2} [V^*]^T [Z_0]^{-1} [V^*]^T [V^*] - \frac{1}{2} [V^*]^T [Z_0]^{-1} [V^*]^T
\]

\[
= 0
\]

Therefore,
\[
[V^*]^T [Z_0]^{-1} [V^*]^T = [V^*]^T [Z_0]^{-1} [V^*]^T - [V^*]^T [S]^T [Z_0]^{-1} [S]^T
\]

\[
\sum_{k=1}^{N} \frac{S_{kj} S_{kj}^*}{Z_{0k}} = \frac{\delta_{ij}}{\sqrt{Z_{0i} Z_{0j}}}
\]

Since
\[
[V^*]^T [Z_0]^{-1} [V^*] = \frac{1}{2} ([Z] + [Z_0])[V^*]
\]

\[
[V^*]^T [Z_0]^{-1} [V^*] = \frac{1}{2} ([Z] - [Z_0])[V^*] = ([Z] - [Z_0]) ([Z] + [Z_0])^{-1} [V^*]
\]

\[
[S]^T = ([Z] - [Z_0]) ([Z] + [Z_0])^{-1} ([Z]^T - [Z_0]^T) = ([Z] + [Z_0]) ([Z]^T - [Z_0]^T)
\]

Also
\[
[S] = ([Z][Z_0]^{-1} + [U])^{-1} ([Z][Z_0]^{-1} - [U])
\]

\[
= ([Z][Z_0]^{-1} + [Z_0])^{-1} ([Z][Z_0]^{-1} - [Z_0])
\]

\[
= [Z_0]^{-1} ([Z][Z_0]^{-1} - [Z][Z_0]^{-1})
\]

Therefore, 
\[
[S] = [Z_0]^{-1} [S]^T [Z_0]^{-1}, \text{ or } [S]^T [Z_0]^{-1} [S]
\]

If reciprocal
\[
Z_0 S_{ij} = Z_{0i} S_{ji}
\]

Example 4.5
Shift in Reference Planes

If at port \( n \), the reference plane is shifted out by a length of \( \ell_n \), the voltage at the reference plane will be

\[
V_n^{/+} = V_n^+ e^{j\theta_n}
\]
\[
V_n^{/-} = V_n^- e^{-j\theta_n}
\]

where \( \theta_n = \beta_n \ell_n \). Let

\[
[\theta] = \begin{bmatrix}
e^{-j\theta_1} & 0 & \cdots & 0 \\
0 & e^{-j\theta_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & e^{-j\theta_N}
\end{bmatrix}
\]

We have

\[
[\theta]^{-1} [V^{/-}] = [S][\theta][V^{/+}]
\]
\[
\Rightarrow [V^{/-}] = [\theta][S][\theta][V^{/+}]
\]
\[
[S'] = [\theta][S][\theta]
\]
Generalized Scattering Parameters

Define the scattering parameters based on the amplitude of the incident and reflected wave normalized to power.

Let

\[ a_n = \frac{V_n^+}{\sqrt{Z_{0n}}} \quad b_n = \frac{V_n^-}{\sqrt{Z_{0n}}} \quad \Rightarrow I_n^+ = \frac{a_n}{\sqrt{Z_{0n}}} \quad I_n^- = \frac{b_n}{\sqrt{Z_{0n}}} \]

thus

\[ P_n^+ = \frac{1}{2} \text{Re}\left\{ V_n^+ I_n^{+\ast} \right\} = \frac{1}{2} \text{Re}\left\{ a_n \sqrt{Z_{0n}} \frac{a_n^*}{\sqrt{Z_{0n}}} \right\} = \frac{1}{2} |a_n|^2 \]

\[ P_n^- = \frac{1}{2} \text{Re}\left\{ V_n^- I_n^{-\ast} \right\} = \frac{1}{2} \text{Re}\left\{ b_n \sqrt{Z_{0n}} \frac{b_n^*}{\sqrt{Z_{0n}}} \right\} = \frac{1}{2} |b_n|^2 \]

The generalized scattering matrix is defined as

\[ [b] = [S'] [a] \]

where

\[ S'_{ij} = \frac{b_i}{a_j} \quad \text{for} \quad a_k = 0 \quad \text{for} \quad k \neq j \]

\[ S'_{ij} = \frac{V_i^-}{\sqrt{Z_{0i}}} \quad S'_{ij} = \frac{V_j^+}{\sqrt{Z_{0j}}} \]

If lossless, \( \sum_{k=1}^{N} S'_{ki} S'_{kj} = \delta_{ij} \) or \( [S'] [S']^\ast = [U] \)

If reciprocal, \( S'_{ij} = S''_{ji} \)
The Transmission (ABCD) Matrix

Define a transmission matrix of a two port network as
\[
V_1 = AV_2 + BI_2
\]
\[
I_1 = CV_2 + DI_2
\]
or in matrix form
\[
\begin{bmatrix}
V_1 \\
I_1 \\
\end{bmatrix} = \begin{bmatrix}
A & B \\
C & D \\
\end{bmatrix} \begin{bmatrix}
V_2 \\
I_2 \\
\end{bmatrix}
\]

Relationship to impedance matrix

\[
A = \frac{V_1}{V_2} \bigg|_{V_2=0} = \frac{Z_{11}}{Z_{21}}
\]

\[
B = \frac{V_1}{I_2} \bigg|_{V_2=0} = \frac{Z_{11}Z_{22} - Z_{12}Z_{21}}{Z_{21}}
\]

\[
C = \frac{I_1}{V_2} \bigg|_{V_2=0} = \frac{1}{Z_{21}}
\]

\[
D = \frac{I_1}{I_2} \bigg|_{V_2=0} = \frac{Z_{22}}{Z_{21}}
\]

If reciprocal, \( AD - BC = 1 \)

Cascading of ABCD matrix:
\[
\begin{bmatrix}
V_1 \\
I_1 \\
\end{bmatrix} = \begin{bmatrix}
A_1 & B_1 \\
C_1 & D_1 \\
\end{bmatrix} \begin{bmatrix}
A_2 & B_2 \\
C_2 & D_2 \\
\end{bmatrix} \begin{bmatrix}
V_3 \\
I_3 \\
\end{bmatrix}
\]

Two-Port Circuits
TABLE 4.1 The $ABCD$ Parameters of Some Useful Two-Port Circuits

<table>
<thead>
<tr>
<th>Circuit</th>
<th>$ABCD$ Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z$</td>
<td>$A = 1$</td>
</tr>
<tr>
<td></td>
<td>$B = Z$</td>
</tr>
<tr>
<td></td>
<td>$C = 0$</td>
</tr>
<tr>
<td></td>
<td>$D = 1$</td>
</tr>
<tr>
<td>$Y$</td>
<td>$A = 1$</td>
</tr>
<tr>
<td></td>
<td>$B = 0$</td>
</tr>
<tr>
<td></td>
<td>$C = Y$</td>
</tr>
<tr>
<td></td>
<td>$D = 1$</td>
</tr>
<tr>
<td>$Z_0, \beta$</td>
<td>$A = \cos \beta$</td>
</tr>
<tr>
<td></td>
<td>$B = jZ_0 \sin \beta$</td>
</tr>
<tr>
<td></td>
<td>$C = jZ_0 \sin \beta$</td>
</tr>
<tr>
<td></td>
<td>$D = \cos \beta$</td>
</tr>
<tr>
<td>$N : 1$</td>
<td>$A = N$</td>
</tr>
<tr>
<td></td>
<td>$B = 0$</td>
</tr>
<tr>
<td></td>
<td>$C = 0$</td>
</tr>
<tr>
<td></td>
<td>$D = \frac{1}{N}$</td>
</tr>
<tr>
<td>$Y_1, Y_2$</td>
<td>$A = 1 + \frac{Y_2}{Y_5}$</td>
</tr>
<tr>
<td></td>
<td>$B = \frac{1}{Y_5}$</td>
</tr>
<tr>
<td></td>
<td>$C = Y_1 + \frac{Y_1 Y_2}{Y_5}$</td>
</tr>
<tr>
<td></td>
<td>$D = 1 + \frac{Y_1}{Y_5}$</td>
</tr>
<tr>
<td>$Z_1, Z_2$</td>
<td>$A = 1 + \frac{Z_1}{Z_3}$</td>
</tr>
<tr>
<td></td>
<td>$B = Z_2 + \frac{Z_1 Z_2}{Z_3}$</td>
</tr>
<tr>
<td></td>
<td>$C = \frac{1}{Z_3}$</td>
</tr>
<tr>
<td></td>
<td>$D = 1 + \frac{Z_2}{Z_3}$</td>
</tr>
</tbody>
</table>

FIGURE 4.13 Equivalent circuits for a reciprocal two-port network. (a) $T$ equivalent. (b) $\pi$ equivalent.

Signal Flow Graphs

Primary Components:
1. Nodes: each port $i$ has two nodes $a_i$ and $b_i$. $a_i$ represents the incident
wave to port \( i \). \( \textbf{b}_i \) represents the reflected wave from port \( i \).

2. Branches: A branch is a directed path between an a-node and a b-node, representing signal flow from node a to node b. Every branch has an associated \( S \) parameter of reflection coefficient.

![Diagram showing a two-port network with branches and reflection coefficients.](image)

**FIGURE 4.14** The signal flow graph representation of a two-port network. (a) Definition of incident and reflected waves. (b) Signal flow graph.

Rules:
1. Series rule
2. Parallel rule
3. Self-loop rule
4. Splitting rule

![Diagrams showing decomposition rules for series, parallel, self-loop, and splitting.](image)

**FIGURE 4.16** Decomposition rules. (a) Series rule. (b) Parallel rule. (c) Self-loop rule. (d) Splitting rule.
Example 4.7

Thru-Reflect Line (TRL) Network Analyzer Calibration

Purpose: to de-embed the effect of the connection between the signal lines of the network analyzer and the actual circuit.
Procedure:
1. Measure the $S$ parameter with direct connection of the two ports of the device under test (DUT).
2. Measure the $S$ parameter with the two ports terminated by loads.
3. Measure the $S$ parameter with the two ports connected by a section of transmission line.

FIGURE 4.20  Block diagram of a network analyzer measurement of a two-port device.

FIGURE 4.21a  Block diagram and signal flow graph for the Thru connection.
Figure 4.21b  Block diagram and signal flow graph for the Reflect connection.
Solving for $S_{11}, S_{12}, S_{22}, \Gamma_L$:

$$S_{11} = T_{11} - S_{22} T_{12}$$
$$S_{22} = T_{12} (1 - S_{22})$$

Using the expressions for $T_{11}$ and $T_{12}$:

$$T_{11} = \frac{b_1}{a_1} \bigg|_{a_2=0} = S_{11} + \frac{S_{22} S_{12}^2}{1 - S_{22}^2}$$
$$T_{12} = \frac{b_1}{a_2} \bigg|_{a_1=0} = \frac{S_{12}^2}{1 - S_{22}^2}$$

$$R_{11} = \frac{b_1}{a_1} \bigg|_{a_2=0} = S_{11} + \frac{S_{12}^2 \Gamma_L}{1 - S_{22} \Gamma_L}$$

$$L_{11} = \frac{b_1}{a_1} \bigg|_{a_2=0} = S_{11} + \frac{S_{22} S_{12}^2 e^{-2\gamma l}}{1 - S_{22} e^{-2\gamma l}}$$
$$L_{12} = \frac{b_1}{a_2} \bigg|_{a_1=0} = \frac{S_{12}^2 e^{-2\gamma l}}{1 - S_{22} e^{-2\gamma l}}$$

Correction: Eq. (4.77a)
\[ L_{21} - L_{21} S_{22} e^{-\gamma l} = T_{21} e^{-\gamma l} - T_{21} S_{22} e^{-\gamma l} \]

Correction: Eq. (4.77b)
\[ T_{11} - S_{22} T_{21} = L_{11} - S_{22} L_{21} e^{-\gamma l} \]
Discontinuities and Modal Analysis (4.6)

Let the modes existing in a waveguide be

\[ E_i^\pm(x,y,z) = \bar{e}_i(x,y) e^{\mp j\beta_i z} \]
\[ H_i^\pm(x,y,z) = \pm \bar{h}_i(x,y) e^{\mp j\beta_i z} \]

Assuming two waveguides \( a \) and \( b \) are connected by an aperture \( S \) located at \( z=0 \). Let the remaining areas at waveguide \( a \) and \( b \) be \( S^a \) and \( S^b \) respectively. Assume only the first mode incident from waveguide \( a \), we have the total tangential fields in \( a \)

\[ E^a(x,y,z) = A_1^+ \bar{e}_1^a(x,y) e^{-j\beta_1^a z} + \sum_{i=1}^{\infty} A_i^+ \bar{e}_i(x,y) e^{j\beta_i^a z} \]
\[ H^a(x,y,z) = A_1^+ \bar{h}_1^a(x,y) e^{-j\beta_1^a z} - \sum_{i=1}^{\infty} A_i^+ \bar{h}_i(x,y) e^{j\beta_i^a z} \]

Likewise in waveguide \( b \)

\[ E^b(x,y,z) = \sum_{i=1}^{\infty} B_i^+ \bar{e}_i^b(x,y) e^{-j\beta_i^b z} \]
\[ H^b(x,y,z) = \sum_{i=1}^{\infty} B_i^+ \bar{h}_i^b(x,y) e^{-j\beta_i^b z} \]

At the aperture \( S \), the fields at both sides must be the same, that is

\[ E^a(x,y,0) = A_1^+ \bar{e}_1^a(x,y) + \sum_{i=1}^{\infty} A_i^+ \bar{e}_i(x,y) = \sum_{i=1}^{\infty} B_i^+ \bar{e}_i^b(x,y) = E^b(x,y,0) \]  \hspace{1cm} (441)
\[ H^a(x,y,0) = A_1^+ \bar{h}_1^a(x,y) - \sum_{i=1}^{\infty} A_i^+ \bar{h}_i(x,y) = \sum_{i=1}^{\infty} B_i^+ \bar{h}_i^b(x,y) = H^b(x,y,0) \]  \hspace{1cm} (442)
And the electric fields at $S^a$ and $S^b$ must equal zero.

Integrate the above electric field equation with the mode pattern of mode $m$ in waveguide $a$ over surface $S+\Sigma^a$, we have

$$\int_{S+\Sigma^a} \left\{ A_1^+ \vec{e}^a_m(x,y) + \sum_{i=1}^{\infty} A_i^- \vec{e}^i(x,y) \right\} ds = \int_{S} \vec{e}^a_m(x,y) \cdot \left\{ \sum_{i=1}^{\infty} B_i^+ \vec{e}^b_i(x,y) \right\} ds$$

Due to the orthogonal properties between the modes in a waveguide, the above equations lead to

$$A_1^+ C_1^a \delta_{m1} + A_m^- C_m^a = \sum_{i=1}^{\infty} q_{mi} B_i^+ (449)$$

where

$$C_m^a = \int_{S+\Sigma^a} \vec{e}^a_m(x,y) \cdot \vec{e}^a_m(x,y) ds$$

$$Q_{mi} = \int_{S} \vec{e}^a_m(x,y) \cdot \vec{e}^b_i(x,y) ds$$

Note that $C_m^a$ is the normalization constant of mode $m$ in waveguide $a$.

Rewriting the above Eq. (188) in matrix form, we have

$$[V] + [C^a] [A^-] = [Q] [B^+] (454)$$

where
Likewise, integrate the magnetic field equation (Eq. 183) with the mode pattern of mode $m$ of waveguide $b$ only over aperture $S$, we have

$$
\int \int_S \left\{ A_1^+ h_1^a(x,y) - \sum_{i=1}^{\infty} A_i^- h_i^a(x,y) \right\} ds
$$

which leads to

$$
A_1^+ p_{m1} - \sum_{i=1}^{\infty} p_{mi} A_i^- = \sum_{i=1}^{\infty} r_{mi} B_i^+(460)
$$

where
Rewriting the above Eq. (198) in matrix form, we have

\[ [I] - [P] [A^-] = [R] [B^+] \] \hspace{1cm} (462)

where

\[
[I] = \begin{bmatrix}
p_{11} \\
p_{21} \\
p_{31} \\
\vdots
\end{bmatrix}, \quad [P]_{mn} = p_{mn}, \quad [R]_{mn} = r_{mn} \hspace{1cm} (463)
\]

From Eq. (193) and Eq. (200), we have

\[
[I] - [P] [A^-] = [R] [Q]^{-1} ([V] + [C^a] [A^-])
\]

\[ \Rightarrow \quad [I] - [R] [Q]^{-1} [V] = ([P] + [R] [Q]^{-1} [C^a]) [A^-] \hspace{1cm} (464) \]

\[ \Rightarrow \quad ([P] + [R] [Q]^{-1} [C^a])^{-1} ([I] - [R] [Q]^{-1} [V]) = [A^-] \]

Thus \([A^-]\) is solved. Using Eq. (193), we have

\[ [Q]^{-1} ([V] + [C^a] [A^-]) = [B^+] \hspace{1cm} (466) \]

Thus \([B^+]\) is solved.
Modal Analysis of an H-Plane Step in Rectangular Waveguide

Assume $TE_{10}$ incident only thus only $TE$ modes reflect in guide 1 and transmit and in guide 2. Then the modes in guide 1 can be specified as

\[
\vec{E}_i^a(x,y) = j\sin \frac{inx}{a}
\]

\[
\vec{H}_i^a(x,y) = -\hat{x} \frac{1}{Z_i} \sin \frac{inx}{a}
\]

\[
Z_i^a = \frac{k_0 \eta_0}{\beta_i^a}
\]

\[
\beta_i^a = \sqrt{k_0^2 - \left(\frac{inx}{a}\right)^2}
\]

and in guide 2

\[
\vec{E}_i^b(x,y) = j\sin \frac{inx}{c}
\]

\[
\vec{H}_i^b(x,y) = -\hat{x} \frac{1}{Z_i} \sin \frac{inx}{c}
\]

\[
Z_i^b = \frac{k_0 \eta_0}{\beta_i^b}
\]

\[
\beta_i^b = \sqrt{k_0^2 - \left(\frac{inx}{c}\right)^2}
\]
Excitation of Waveguides (4.7)

Assume sources \( \vec{J} \) and \( \vec{M} \) exist in a waveguide between \( z_1 \) and \( z_2 \). The tangential fields outside this region can be expressed as

\[
\vec{E}^+(x,y,z) = \sum_{i=1}^{\infty} A_i^+ \vec{E}_n^+ = \sum_{i=1}^{\infty} A_i^+ (\vec{e}_i(x,y) + \hat{z} e_{zn}) e^{-j\beta_i^+ z}, \quad z > z_2
\]

\[
\vec{H}^+(x,y,z) = \sum_{i=1}^{\infty} A_i^+ \vec{H}_n^+ = \sum_{i=1}^{\infty} A_i^+ (\vec{h}_i(x,y) + \hat{z} h_{zn}) e^{-j\beta_i^+ z}, \quad z > z_2
\]

\[
\vec{E}^-(x,y,z) = \sum_{i=1}^{\infty} A_i^- \vec{E}_n^- = \sum_{i=1}^{\infty} A_i^- (\vec{e}_i(x,y) - \hat{z} e_{zn}) e^{j\beta_i^- z}, \quad z < z_1
\]

\[
\vec{H}^-(x,y,z) = \sum_{i=1}^{\infty} A_i^- \vec{H}_n^- = -\sum_{i=1}^{\infty} A_i^- (\vec{h}_i(x,y) - \hat{z} h_{zn}) e^{j\beta_i^- z}, \quad z < z_1
\]

Assume \( \vec{M} = 0 \), from reciprocity theorem, we have

\[
\oint_S (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1) \cdot d\vec{s} = \int_V (\vec{E}_2 \times \vec{J}_1 - \vec{E}_1 \times \vec{J}_2) \, dv
\]

Let \( \vec{J}_1 = \vec{J} \), then \( \vec{E}_1 \) and \( \vec{H}_1 \) are the fields generated by \( \vec{J} \), which are \( \vec{E}^+ \), \( \vec{H}^+ \), \( \vec{E}^- \) and \( \vec{H}^- \).

Let \( \vec{E}_2 = \vec{E}_n^- \) and \( \vec{H}_2 = \vec{H}_n^- \), we have
\[ \int_{z=z_2} \left( \vec{E}^+ \times \vec{H}^- - \vec{E}^- \times \vec{H}^+ \right) \cdot \hat{z} \, ds + \int_{z=z_1} \left( \vec{E}^- \times \vec{H}^- - \vec{E}^- \times \vec{H}^+ \right) \cdot (-\hat{z}) \, ds = \int_{V} \vec{E}_n^- \cdot \vec{J} \, dv \]

\[ \Rightarrow A^+_n = -\frac{\int_{V} (\vec{e}_n^- \times \vec{h}_n^-) e^{j\beta_n z} \cdot \vec{J} \, dv}{2 \int_{V} \vec{e}_n \times \vec{h}_n \, ds} \]

Likewise, let \( \vec{E}_2 = \vec{E}_n^+ \) and \( \vec{H}_2 = \vec{H}_n^+ \), we have

\[ \int_{z=z_2} \left( \vec{E}^+ \times \vec{H}^+ - \vec{E}^+ \times \vec{H}^+ \right) \cdot \hat{z} \, ds + \int_{z=z_1} \left( \vec{E}^- \times \vec{H}^- - \vec{E}^- \times \vec{H}^- \right) \cdot (-\hat{z}) \, ds = \int_{V} \vec{E}_n^+ \cdot \vec{J} \, dv \]

\[ \Rightarrow A^-_n = -\frac{\int_{V} (\vec{e}_n^+ \times \vec{h}_n^+) \, ds}{2 \int_{V} \vec{e}_n \times \vec{h}_n \, ds} \]
Probe-Fed Rectangular Waveguide

\[ J(x,y,z) = I_0 \delta(x - \frac{a}{2}) \delta(z) \delta y \] for \( 0 \leq y \leq b \)
Electromagnetic Theorems (1.3, 1.9)

Boundary conditions

Let the fields in media 1 denoted by subscript 1 and media 2 subscript 2. At the boundary of media 1 and 2, the electromagnetic fields satisfy the following conditions.

\[-\hat{n} \times (\vec{E}_2 - \vec{E}_1) = \vec{M}_s\]
\[\hat{n} \cdot (\vec{D}_2 - \vec{D}_1) = \rho_s\]
\[\hat{n} \times (\vec{H}_2 - \vec{H}_1) = \vec{J}_s\]
\[\hat{n} \cdot (\vec{B}_2 - \vec{B}_1) = \rho_{ms}\]

where $\hat{n}$ points from media 1 to 2, $\vec{M}_s$ the magnetic surface current, $\vec{J}_s$ the electric surface current, $\rho_{ms}$ the magnetic surface charge, $\rho_s$ the electric surface charge.

Uniqueness Theorem

In a region bounded by a close surface $S$, if two sets of electromagnetic fields satisfy the following conditions:
1. The sources in the region are the same.
2. The tangential electric fields or the tangential electric fields on the boundary are the same.

Then, these two sets of electromagnetic fields are the same everywhere in the region.

Equivalence Principle

In a region bounded by a close surface $S$, let the field on $S$ be $\vec{E}^a$ and $\vec{H}^a$. If the exterior fields are replaced with $\vec{E}^b$ and $\vec{H}^b$, then the interior fields will be the same if $\vec{M}_s = -\hat{n} \times (\vec{E}_b - \vec{E}_a)$ and $\vec{J}_s = \hat{n} \times (\vec{H}_b - \vec{H}_a)$ are placed on surface $S$.

Examples: PEC boundary, PMC boundary.
**Image Theory**

In front of a planar PEC, the fields are the same if the PEC is removed and the images of the sources are placed at the other side. For an electric charge, the image is the negative of the charge. For an magnetic charge, the image is the same charge.

For the case of PMC, the image of an electric charge is the same, while the image of an magnetic charge is the negative.

**Reciprocity Theorem**

For two sets electromagnetic fields generated by sources \((\vec{J}_1, \vec{M}_1)\) and \((\vec{J}_2, \vec{M}_2)\) in the same space bounded by surface \(S\), we have

\[
\oint_{\partial S} (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1) \cdot d\vec{s} = \int_{\nu} (\vec{E}_2 \cdot \vec{J}_1 - \vec{E}_1 \cdot \vec{J}_2 + \vec{H}_1 \cdot \vec{M}_2 - \vec{H}_2 \cdot \vec{M}_1) \, dv
\]
Impedance Matching and Tuning (5)

Smith Charts (2.4)

Let $Z_0$ be characteristic impedance of a transmission line. For a load $Z_L$, let

$$z_L = \frac{Z_L}{Z_0}$$

be the normalized impedance, then the reflection coefficient $\Gamma$ becomes

$$\Gamma = \frac{z_L - 1}{z_L + 1}$$

Let $\Gamma = \Gamma_r + j\Gamma_i$ and $z_L = r_L + jx_L$, we have

$$\left( \Gamma_r - \frac{r_L}{1+r_L} \right)^2 + \Gamma_i^2 = \left( \frac{1}{1+r_L} \right)^2$$

$$\left( \Gamma_r - 1 \right)^2 + \left( \Gamma_i - \frac{1}{x_L} \right)^2 = \left( \frac{1}{x_L} \right)^2$$

These define the constant resistance and reactance curves.

Similarly

$$\Gamma = \frac{1}{y_L} - 1 = y_L - \frac{1}{y_L+1} = \frac{y_L - 1}{y_L+1}$$

Thus we can conclude that the constant conductance and susceptance curves are the same forms as the constant resistance and reactance curves.

To sum up,

1. Smith chart is a plot of the reflection coefficient $\Gamma$ on the complex plane with constant resistance and reactance curves overlapped. That is the real part of $\Gamma$ is plotted as the x coordinate, the imaginary part the y coordinate.
2. The $\Gamma$ at a distance $\ell$ from the load is a clockwise rotation of angle $2\beta\ell$, where $\beta$ is the propagation constant of the transmission line.

3. The constant resistance and reactance curves can be used for admittance except that the Smith chart becomes a plot of $-\Gamma$.

4. The admittance value can be read from Smith chart by rotating $180^\circ$.

Example 2.4
Matching with Lumped Elements (L Networks)

(a)

(b)

Analytic Solutions

(a) \( R_L > Z_0 \)

\[
B = \frac{X_L \pm \sqrt{R_L/Z_0}}{R_L + X_L^2 - Z_0 R_L} \]

\[
X = \frac{1}{B} + \frac{X_L Z_0}{R_L} - \frac{Z_0}{BR_L}
\]

(b) \( R_L < Z_0 \)

\[
B = \pm \frac{\sqrt{(Z_0 - R_L)/R_L}}{Z_0}, \quad X = \pm \sqrt{R_L (Z_0 - R_L)} - X_L
\]

Smith Chart Solutions

1. \( R_L > Z_0 \). Use (a)
   a. Convert to admittance plot.
   b. Move along constant conductance curve until intercept with the constant resistance curve equal to 1.
   c. Convert back to impedance plot.
   d. Find the required reactance.
2. \( R_L < Z_0 \). Use (b)
   a. Move along constant resistance curve until intercept with the constant admittance curve equal to 1.
   b. Convert to admittance plot.
   c. Find the required susceptance.

Example 5.1
Figure 5.3 Continued. (b) The two possible L-section matching circuits. (c) Reflection coefficient magnitudes versus frequency for the matching circuits of (b).
**FIGURE 5.3** Solution to Example 5.1. (a) Smith chart for the L-section matching networks.

Transmission line can provide a series capacitance up to about 0.5 pF. Greater values (up to about 25 pF) can be obtained using a metal-insulator-metal (MIM) sandwich, either in monolithic or chip (hybrid) form.
Single-Stub Tuning (5.2)

Analytic Solutions

1. Shunt Stubs

\[ t = \frac{X_L \pm \sqrt{R_L \left(Z_0 - R_L\right)^2 + X_L^2}}{R_L - Z_0} \]

\[ \frac{d}{\lambda} = \begin{cases} \frac{1}{2\pi} \tan^{-1} t, & \text{for } t \geq 0 \\ \frac{1}{2\pi} \left(\pi + \tan^{-1} t\right), & \text{for } t < 0 \end{cases} \]

Open stub: \[ I_0 = -\frac{1}{\lambda} \tan^{-1} \frac{B}{Y_0} \]
Short stub: \[ I_s = \frac{1}{\lambda} \tan^{-1} \frac{Y_0}{B} \]

Where \[ B = \frac{R_L^2 t - (Z_0 - tX_L)(X_L + tZ_0)}{Z_0[R_L^2 + (X_L + Z_0)^2]} \]

2. Series Stubs
Shunt (Series) Stubs
1. Use admittance (impedance) plot.
2. Rotate clockwise along constant $\Gamma$ curve until intercept with the constant conductance (resistance) curve of value 1.
3. Compensate the remaining susceptance (reactance) by a suitable length of open or short stub.

where $x = \frac{G_L^2 t - (Y_0 - tB_L)(B_L + tY_0)}{Y_0\left[G_L^2 + (B_L + Y_0t)^2\right]}$
Example 5.2

(a) Solution #1
(b) Solution #2
(c) Reflection coefficient magnitudes versus frequency for the tuning circuits of (b).

FIGURE 5.5 Continued. (b) The two shunt-stub tuning solutions. (c) Reflection coefficient magnitudes versus frequency for the tuning circuits of (b).

FIGURE 5.5 Solution to Example 5.2. (a) Smith chart for the shunt-stub tuners.
Double-Stub Tuning (5.3)

Analytical Solution

\[
B_1 = -B_L + \frac{Y_0 \pm \sqrt{(1 + t^2)G_L Y_0 - G_L^2 t^2}}{t}
\]

\[
B_2 = \frac{\pm Y_0 \sqrt{(1 + t^2)G_L Y_0 - G_L^2 t^2} + G_L Y_0}{G_L t}
\]

\[t = \tan \beta d\]

Requirement:

\[0 \leq G_L \leq Y_0 \frac{1 + t^2}{t^2} = \frac{Y_0}{\sin^2 \beta d}\]

Smith Chart Solutions

1. Use admittance plot.
2. Rotate the constant conductance circle of value 1 counterclockwise by a distance \(d\).
3. Move \(Y_L\) along the constant conductance curve until intercepting the rotated circle in 2. The difference of the susceptance determines the length of the stub 2.
4. Rotate the intercepting point back to constant conductance circle of value 1. The susceptance value determine the length of stub 1.

**FIGURE 5.8** Smith chart diagram for the operation of a double-stub tuner.

**Example 5.4**

\[ Z_0 = 50\Omega, \quad Z_L = 60 - j80\Omega, \quad d = \frac{\lambda}{8} \]
FIGURE 5.9 Solution to Example 5.4. (a) Smith chart for the double-stub tuners.

Solution 1

Solution 2

FIGURE 5.9 Continued. (b) The two double-stub tuning solutions. (c) Reflection coefficient magnitudes versus frequency for the tuning circuits of (b).
Transformers (5.4 – 5.9)

Quarter-Wave Transformer

Match a real load \( Z_L \) to \( Z_0 \) by a section of transmission line with characteristic impedance \( Z_1 \) and length \( \ell \).

\[
Z_{in} = \frac{Z_L + jZ_1 \tan \beta \ell}{Z_1 + jZ_L \tan \beta \ell} \Rightarrow Z_1 = \sqrt{Z_0 Z_L}, \quad \ell = \lambda/4
\]

The reflection coefficient becomes

\[
\Gamma = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} = \frac{Z_1(Z_L - Z_0) + j\tan \beta \ell(Z_1^2 - Z_0 Z_L)}{Z_1(Z_L + Z_0) + j\tan \beta \ell(Z_1^2 + Z_0 Z_L)}
\]

\[
= \frac{Z_L - Z_0}{Z_L + Z_0 + j2\tan \beta \ell \sqrt{Z_0 Z_L}}.
\]

\[
|\Gamma| = \frac{|Z_L - Z_0|}{\sqrt{Z_L + Z_0 + j2\tan \beta \ell \sqrt{Z_0 Z_L}}}
\]

\[
= \frac{1}{\left(1 + \frac{4Z_0 Z_L}{(Z_L - Z_0)^2} \sec^2 \beta \ell\right)^{1/2}}
\]

\[
= \frac{|Z_L - Z_0|}{2\sqrt{Z_0 Z_L}} |\cos \theta|, \quad \theta = \beta \ell \approx \frac{\pi}{2}
\]

for a given \( \Gamma_m = |\Gamma| \), solve for \( \theta_m \), we have
\[
\cos \theta_m = \frac{\Gamma_m}{\sqrt{1 - \Gamma_m^2}} \frac{2 \sqrt{Z_0 Z_L}}{|Z_L - Z_0|}
\]

Assume TEM mode,

\[\theta = \beta l = \frac{\pi f}{2 f_0} \Rightarrow f_m = \frac{2 \theta_m f_0}{\pi}\]

The bandwidth becomes

\[
\frac{\Delta f}{f_0} = \frac{2(f_0 - f_m)}{f_0} = 2 - \frac{4 \theta_m}{\pi} = 2 - \frac{4}{\pi} \cos^{-1} \left( \frac{\Gamma_m}{\sqrt{1 - \Gamma_m^2}} \frac{2 \sqrt{Z_0 Z_L}}{|Z_L - Z_0|} \right)
\]

**FIGURE 5.12** Reflection coefficient magnitude versus frequency for a single-section quarter matching transformer with various load mismatches.

**Example 5.5**

\[Z_0 = 50 \, \Omega, \ Z_L = 10 \, \Omega, \ f_0 = 3 \, \text{GHz}, \ \text{SWR} \leq 1.5.\]

\[
\frac{\Delta f}{f_0} = 29\%\]
Theory of Small Reflections

A multisection transformer consists of \( N \) equal-length sections of transmission lines. Let

\[
\Gamma_0 = \frac{Z_1 - Z_0}{Z_1 + Z_0}, \quad \Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n}, \quad \Gamma_N = \frac{Z_L - Z_N}{Z_L + Z_N}, \quad \theta = \beta \ell
\]

Assume that the reflection coefficients at each junction is very small, the total reflection coefficient can be approximated by
\[ \Gamma(\theta) = \Gamma_0 + \Gamma_1 e^{-2j\theta} + \Gamma_2 e^{-4j\theta} + \cdots + \Gamma_N e^{-2jN\theta} \]

If \( n \) is symmetrical, that is, \( \Gamma_0 = \Gamma_N \), \( \Gamma_1 = \Gamma_{N-1} \), \( \Gamma_2 = \Gamma_{N-2} \), etc. Then, \[
\Gamma(\theta) = e^{-jN\theta} \left\{ \Gamma_0 \left[ e^{jN\theta} + e^{-jN\theta} \right] + \Gamma_1 \left[ e^{j(N-2)\theta} + e^{-j(N-2)\theta} \right] + \cdots \right\}
\]

If \( N \) is even, the previous equation becomes
\[
\Gamma(\theta) = 2e^{-jN\theta} \left\{ \Gamma_0 \cos N\theta + \Gamma_1 \cos (N-2)\theta + \cdots \right. \\
\left. + \Gamma_n \cos (N-2n)\theta + \cdots + \frac{1}{2} \Gamma_{N/2}\right\}
\]

If \( N \) is odd
\[
\Gamma(\theta) = 2e^{-jN\theta} \left\{ \Gamma_0 \cos N\theta + \Gamma_1 \cos (N-2)\theta + \cdots \right. \\
\left. + \Gamma_n \cos (N-2n)\theta + \cdots + \Gamma_{(N-1)/2} \cos \theta \right\}
\]

**Binomial Multisection Matching Transformers**

Let
\[
\Gamma(\theta) = A \left( 1 + e^{-2j\theta} \right)^N
\]
and the length of each section equals the quarter wavelength at the center frequency. That is \( \theta = \frac{\pi}{2} \).

We have
\[
\Gamma(\theta) = A \left( C_0^N + C_1^N e^{-2j\theta} + C_2^N e^{-4j\theta} + \cdots + C_N^N e^{-2Nj\theta} \right)
\]

Thus
\[
\Gamma_n = AC_n^N
\]

Property: flat near the center frequency

Proof:

For \( n < N \)
Thus, at \( \theta = \frac{\pi}{2} \),

\[
\frac{d^n \Gamma(\theta)}{d\theta^n} \bigg|_{\theta = \frac{\pi}{2}} = 0
\]

When frequency approaching zero, the electrical length of each section also approaching zero. We have

\[
\Gamma(0) = 2^N A = \frac{Z_L - Z_0}{Z_L + Z_0} \quad \Rightarrow \quad A = 2^{-N} \frac{Z_L - Z_0}{Z_L + Z_0}
\]

The above result is not rigorous, since the limit only holds when multiple reflections are considered.

Since \( A \) is known, every \( \Gamma_n \) can be computed. Also all the required \( Z_n \) can computed from \( Z_0 \) or \( Z_L \).

Bandwidth: Let \( \Gamma_m \) be the maximum value of reflection coefficient that can be tolerated over the passband. Let \( \theta_m \) be the correspondong \( \theta \) value at the lower edge. That is \( \theta_m < \frac{\pi}{2} \). We have

\[
\Gamma_m = |\Gamma(\theta_m)| = |A||e^{-j\theta_m}2\cos\theta_m|^N = 2^N|A|\cos^N\theta_m
\]

Thus

\[
\theta_m = \cos^{-1}\left[ \frac{1}{2} \left( \frac{\Gamma_m}{|A|} \right)^{\frac{1}{N}} \right] \quad \Rightarrow \quad \Delta f = \frac{4\theta_m}{\pi^2} = 2 - \frac{4}{\pi} \cos^{-1}\left[ \frac{1}{2} \left( \frac{\Gamma_m}{|A|} \right)^{\frac{1}{N}} \right]
\]

To sum up,
1. From \( Z_0 \), \( Z_L \) and \( N \), find \( A \) by using Eq. 5.49.

2. From \( A \) and the given \( \Gamma_m \) find the bandwidth by using Eq. 5.55.

3. If the bandwidth is not satisfied, increase \( N \) and repeat 1 and 2.

4. Find \( \Gamma_n \) by Table 5.1 or Eq. 5.53, or the relationship

\[
\frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n} = \Gamma_n = AC_n^N
\]
Example 5.6

![Graph of reflection coefficient magnitude versus frequency for multisection binomial matching transformers of Example 5.6. $Z_L = 50 \, \Omega$ and $Z_0 = 100 \, \Omega$.](image)

Chebyshev Multisection Matching Transformer

Chebyshev Polynomials $T_n(x)$

$T_1(x) = x$

$T_2(x) = 2x^2 - 1$

$T_3(x) = 4x^3 - 3x$

$T_4(x) = 8x^4 - 8x^2 + 1$

$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$

Characteristics:

1. $|x| \leq 1 \Rightarrow |T_n(x)| \leq 1$.
   
   $|T_n(\pm 1)| = 1$

2. $|x| > 1 \Rightarrow |T_n(x)| > 1$

3. For $|x| > 1$, $|T_n(x)|$ increases faster with $x$ as $n$ increases.

4. $T_n(\cos \theta) = \cos n \theta$

Suppose the passband is $\theta_m < \theta < \pi - \theta_m$. Let
\[
\Gamma(\theta) = 2e^{-jN\theta}\left\{ \Gamma_0 \cos N\theta + \Gamma_1 \cos (N-2)\theta + \cdots + \Gamma_n \cos (N-2n)\theta + \cdots \right\}
\]

\[
= Ae^{-jN\theta} T_N\left(\frac{\cos \theta}{\cos \theta_m}\right)
\]

Since \(\frac{\cos \theta}{\cos \theta_m}\leq 1\) in the passband \(\theta_m < \theta < \pi - \theta_m\), \(|\Gamma(\theta)| \leq |A|\) in this range

and

\[
\Gamma_m = |\Gamma(\theta_m)| = |A|.
\]

Similar to previous section

\[
\Gamma(0) = \frac{Z_L - Z_0}{Z_L + Z_0} = AT_n\left(\frac{\cos \theta}{\cos \theta_m}\right) = AT_n\left(\frac{1}{\cos \theta_m}\right)
\]

Combine the previous two equations, we have

\[
\left|\frac{Z_L - Z_0}{Z_L + Z_0}\right| = T_N\left(\frac{1}{\cos \theta_m}\right) = T_N\left(\frac{1}{\cos \theta_m}\right)
\]

\[
\Rightarrow \frac{1}{\cos \theta_m} = \cosh\left[\frac{1}{N} \cosh^{-1}\left(\frac{1}{\Gamma_m}\left|\frac{Z_L - Z_0}{Z_L + Z_0}\right|\right)\right]
\]

To sum up
1. From the given \(\Gamma_m, Z_L, Z_0,\) and \(N\), find \(\theta_m\) by using Eq. 5.63.
2. Determine the bandwidth by using Eq. 5.64.
3. If the bandwidth is not satisfied, increase \(N\) and repeat 1 and 2.
4. From Eq. 5.62, decide \(A\). By Eq. 5.61, all the \(\Gamma_n\) can be found. \(Z_n\) can be determined from \(\Gamma_n\) or by looking up Table 5.2.

Example 5.7
Tapered Lines

Let the characteristic impedance of a section of transmission line with length $L$ be a function of $z$, that is $Z=Z(z)$. By approximating $Z(z)$ with stair case functions, using small reflection formula, we have

$$\Delta \Gamma = \frac{(Z+\Delta Z)-Z}{Z+\Delta Z} \approx \frac{\Delta Z}{2Z}$$

In the limit as $\Delta Z \to 0$, we have the exact differential

$$d\Gamma = \frac{d}{dz} \left( \frac{\ln Z}{Z_0} \right) \Rightarrow \Gamma(\theta) = \frac{1}{2} \int_0^L e^{-2j\beta z} \frac{d}{dz} \left( \frac{\ln Z}{Z_0} \right) dz$$

Exponential Taper

Let

$$Z(z) = Z_0 e^{az}$$

$$a = \frac{1}{L} \ln \frac{Z_L}{Z_0}$$

Then
\[
\Gamma(\theta) = \frac{1}{2} \int_0^L e^{-2\beta z} \frac{d}{dz} (\ln e^{-\theta}) dz
\]

\[
= \ln \frac{Z_L}{Z_0} L \\
= \frac{1}{2L} \int_0^L e^{-2\beta z} dz
\]

\[
= \ln \frac{Z_L}{Z_0} e^{-\beta L \sin \beta L} \frac{1}{\beta L}
\]

**FIGURE 5.19**  A matching section with an exponential impedance taper. (a) Variation of impedance. (b) Resulting reflection coefficient magnitude response.

Note: peaks in $|\Gamma|$ decrease with increasing length. The length should be greater than $\frac{\lambda}{2}$ to minimize the mismatch at low frequencies.

**Triangular Taper**

Let
Note: for $\beta L > 2\pi$, the peaks is larger than the corresponding peaks of the exponential case. The first null occurs at $\beta L = 2\pi$

\[
Z(z) = \begin{cases} 
Z_0 e^{2(z/L)^2 \ln \frac{Z_L}{Z_0}} & \text{for } 0 \leq z \leq \frac{L}{2} \\
Z_0 e^{(\frac{4z^2}{L^2} - 2z^2/L^2 - 1) \ln \frac{Z_L}{Z_0}} & \text{for } \frac{L}{2} \leq z \leq L
\end{cases}
\]

\[
\Gamma(\theta) = \frac{1}{2} e^{-j\beta L \ln \left[ \frac{Z_L}{Z_0} \right]} \left[ \frac{\sin \frac{\beta L}{2}}{\frac{\beta L}{2}} \right]^2
\]

Klopfenstein Taper

Reflection coefficient is minimum over the passband, or the length of the matching section is shortest for a maximum reflection coefficient specified over the passband. Let

\[
\ln Z(z) = \frac{1}{2} \ln Z_0 Z_L + \Gamma_0 \frac{A}{\cosh A} \frac{2 \left( \frac{z}{L} - 1, A \right)}{\cosh A}, \text{ for } 0 \leq z \leq L
\]

where
and \( I_1(x) \) is the modified Bessel function. Then,

\[
\Gamma(\theta) = \Gamma_0 e^{-\beta L \cos \left( \frac{\beta L}{\cos A} \right)}
\]

where \( \Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} \). Define the passband as when \( \beta L > A \). \( \Gamma(\theta) \) is equal ripple in passband. Then

\[
\Gamma_m = \frac{\Gamma_0}{\cosh A}
\]

\( \Gamma(\theta) \) oscillates between \( \pm \frac{\Gamma_0}{\cosh A} \) for \( \beta L > A \)

Example 5-8

The Klopfenstein taper is seen to give the desired response of \( \Gamma \leq \Gamma_m = 0.02 \) for \( \beta L \geq 1.13 \pi \), which is lower than either the triangular or exponential taper responses.
The Bode-Fano Criterion

**Theoretical Limits**

1. **Bode-Fano criterion** gives for certain canonical types of load impedances a theoretical limit on the minimum reflection coefficient magnitude that can be obtained with an arbitrary matching network.

2. For a given load, a broader bandwidth can be achieved only at the expense of a higher reflection coefficient in the passband.

3. The passband reflection coefficient $\Gamma_m$ cannot be zero unless $\Delta \omega = 0$.

4. As $R$ and/or $C$ increases, the quality of match must decrease. Thus, higher-Q circuits are intrinsically harder to match than are lower Q circuits.
Microwave Resonators (6)

What is resonance?
1. The natural modes of a system.
   a. Metallic cavity.
   b. A long beam.
   c. Musical instrument.
   d. LC circuit.
2. Self-sustained if lossless.
3. Energy grows to infinity if fed by a source which has a spectrum containing the resonant frequency if lossless.

Quality Factor

\[ Q = \frac{2\pi \text{average energy stored}}{\text{energy loss in one period}} \]

Series and Parallel Resonant Circuits (6.1)

Series Resonant Circuit

\[ Z_{in} = R + j\omega L + \frac{1}{j\omega C} \]
\[ P_{in} = P_{loss} + 2j\omega(W_m - W_e) \]

Power Loss: \( P_{loss} = \frac{1}{2} |I|^2 R \)

Average Stored Magnetic Energy: \( W_m = \frac{1}{4} |I|^2 L \)
Average Stored Electric Energy: \[ W_e = \frac{1}{4} \left| V \right|^2 \frac{1}{\omega^2 C} \]

Resonant Frequency: \[ \omega_0 = \frac{1}{\sqrt{LC}} \]

Quality Factor: \[ Q = 2\pi \frac{W_m + W_e}{T \cdot P_{\text{loss}}} = \omega \frac{W_m + W_e}{P_{\text{loss}}} \]

At \( \omega = \omega_0 \), \( Q = \omega_0 \frac{2W_m}{P_{\text{loss}}} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 R C} \)

Near resonance, let \( \omega = \omega_0 + \Delta \omega \)

\[
Z_{in} = R + j\omega L \left( 1 - \frac{1}{\omega^2 LC} \right) = R + j\omega L \left( \frac{\omega^2 - \omega_0^2}{\omega^2} \right) = R + j2L\Delta \omega = R + j\frac{2RQ\Delta \omega}{\omega_0}
\]

Let complex resonant frequency be \( \omega'_0 = \omega_0 \left( 1 + \frac{j}{2Q} \right) \)

Treat the circuit as lossless, and use the complex resonant frequency to account for the loss

\[
Z_{in} = j2L(\omega - \omega'_0) = \frac{\omega_0 L}{Q} + j2L(\omega - \omega_0) = R + j2L\Delta \omega
\]

\[ \omega'_0 = \omega_0 \left( 1 + \frac{j}{2Q} \right) \]

Half-power bandwidth \( BW = \frac{1}{Q} \)
Parallel Resonant Circuit

\[ Q = \omega_0 \frac{2W_m}{P_{loss}} = \frac{R}{\omega_0 L} = \omega_0 RC \]

Similarly,

\[ Z_{in} = \left( \frac{1}{R} + \frac{1 - \Delta \omega / \omega_0}{j \omega_0 L} + j \omega_0 C + j \Delta \omega C \right)^{-1} \]

\[ = \left( \frac{1}{R} + j \frac{\Delta \omega}{\omega_0^2 L} + j \Delta \omega C \right)^{-1} \]

\[ = \left( \frac{1}{R} + 2 j \Delta \omega C \right)^{-1} \]

\[ = \frac{R}{1 + 2 j \Delta \omega RC} = \frac{R}{1 + 2 j Q \Delta \omega / \omega_0} \]

\[ BW = \frac{1}{Q} \]

Loaded and Unloaded Q
Define the Q of an external load $R_L$ as $Q_e$, then

$$Q_e = \begin{cases} \frac{\omega L}{R_L} & \text{series} \\
\frac{R_L}{\omega L} & \text{parallel} \end{cases}$$

The loaded Q can be expressed as

$$\frac{1}{Q_L} = \frac{1}{Q_e} + \frac{1}{Q}$$

**Transmission Line Resonators (6.2)**

**Short-Circuited $\lambda/2$ Line**

$$Z_{in} = Z_0 \tanh(\alpha + j\beta) = Z_0 \frac{\tanh \alpha \ell + j\tan \beta \ell}{1 + j\tan \beta \ell \tanh \alpha \ell}$$

Assume small loss, $\tanh \alpha \ell \approx \alpha \ell$

Assume a TEM line,

$$\beta \ell = \frac{\omega \ell}{v_p} = \frac{\omega_0 \ell}{v_p} + \frac{\Delta \omega \ell}{v_p}$$

Since $\ell = \lambda/2$ at $\omega = \omega_0$,

$$\beta \ell = \pi + \frac{\Delta \omega \pi}{\omega_0}$$

and

$$\tan \beta \ell = \tan(\pi + \frac{\Delta \omega \pi}{\omega_0}) = \tan \frac{\Delta \omega \pi}{\omega_0} \approx \frac{\Delta \omega \pi}{\omega_0}$$

Thus

$$Z_{in} = Z_0 (\alpha \ell + j \frac{\Delta \omega \pi}{\omega_0})$$

Similar to a series RLC circuit,

$$R = Z_0 \alpha \ell$$

$$L = Z_0 \frac{\pi}{2 \omega_0}$$
\[ C = \frac{1}{\omega_0^2 L} \]
\[ Q = \frac{\beta}{2\alpha} \]

**Short-Circuited \( \lambda/4 \) Line**

\[ Z_{in} = Z_0 \tanh (\alpha + j\beta)\ell \approx \frac{Z_0}{\alpha\ell + j\pi \frac{\Delta\omega}{\omega_0}} \]

Similar to a parallel RLC circuit,

\[ R = \frac{Z_0}{\alpha\ell} \]
\[ C = \frac{\pi}{4\omega_0 Z_0} \]
\[ L = \frac{1}{\omega_0^2 C} \]
\[ Q = \frac{\beta}{2\alpha} \]

**Open-Circuited \( \lambda/2 \) Line**

Similar to a parallel RLC circuit,

\[ Z_{in} = Z_0 \coth (\alpha + j\beta)\ell \approx \frac{Z_0}{\alpha\ell + j\pi \frac{\Delta\omega}{\omega_0}} \]

\[ R = \frac{Z_0}{\alpha\ell} \]
\[ C = \frac{\pi}{2\omega_0 Z_0} \]
Rectangular Waveguide Cavities (6.3)

Cutoff wavenumber

\[ k_{mn} = \sqrt{\left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 + \left( \frac{\ell\pi}{d} \right)^2} \]

Resonant Frequency of mode \( TE_{mn} \) or \( TM_{mn} \)

\[ f_{mn} = \frac{c}{2\sqrt{\varepsilon_r\mu_r}} \sqrt{\left( \frac{m}{a} \right)^2 + \left( \frac{n}{b} \right)^2 + \left( \frac{\ell}{d} \right)^2} \]

Circular Waveguide Cavities (6.4)

Dielectric Resonators (6.5)

Fabry-Perot Resonators (6.6)

Excitation of Resonators (6.7)

Critical Coupling → matching, maximum power

\[ Z_0 = R \]
\[ Q_e = Q \]

Define coefficient coupling, \( g = \frac{Q}{Q_e} \)

\( g < 1 \) : undercoupled to the feedline
\( g = 1 \) : critically coupled to the feedline
\( g > 1 \) : overcoupled to the feedline

A Gap-Coupled Microstrip Resonator
where 

\[ z = \frac{Z}{Z_0} = -j \frac{1}{\omega C} + Z_0 \cot \beta \ell = -j \frac{\tan \beta \ell + b_c}{b_c \tan \beta \ell} \]

where \( b_c = Z_0 \omega C \).

Condition of resonance:

\[ z = 0 \Rightarrow \tan \beta \ell + b_c = 0 \Rightarrow \tan \omega \frac{\ell}{v_p} + \omega C Z_0 = 0 \]

which is a function of \( \omega \).

Note: assume ideal transmission line such that \( \beta = \frac{\omega}{v_p} \)

Characteristic near resonance

By Taylor’s expansion near resonant frequency \( \omega_1 \)

\[ z(\omega) = z(\omega_1) + (\omega - \omega_1) \left[ \frac{dz(\omega)}{d\omega} \right]_{\omega_1} + \cdots \]

First,
So, \[
\frac{dz(\omega)}{d\omega} \bigg|_{\omega_0} = \frac{-j\sec^2 \beta \ell}{b_c \tan \beta \ell} \frac{d(\beta \ell)}{d\omega} = \frac{j(1+b_c^2)}{b_c^2 v_p} \frac{\ell}{v_c^2 v_p} \approx \frac{j \ell}{\omega v_c^2} = \frac{j\pi}{\omega_1 b_c^2}
\]

So,
\[
z(\omega) = (\omega - \omega_1) \frac{j\pi}{\omega_1 b_c^2}
\]

Compare to series RLC circuit \[Z_{in} = R + 2jL\Delta\omega,\]
\[
L = \frac{\pi}{2\omega_1 b_c^2}
\]

Using complex frequency \[\omega_1' = \omega_1 (1 + \frac{j}{2Q})\] to include the effect of loss, we have
\[
z(\omega) = \frac{\pi}{2Q b_c^2} + j(\omega - \omega_1) \frac{\pi}{\omega_1 b_c^2}
\]

where \(Q\) is approximated by the \(Q\) of the open-circuit \(\frac{\lambda}{2}\) transmission line since the gap capacitance is very small. For critical coupling
\[
b_c = \sqrt{\frac{\pi}{2Q}}
\]

**Example 6.6**

**An Aperture-Coupled Cavity**

\[
y = Z_0 y = -j \frac{X_L}{Z_0} = -j \frac{\tan \beta \ell + x_L}{x_L \tan \beta \ell}
\]
where \( x_L = \frac{\omega L}{Z_0} \), similarly to previous section,

\[
y(\omega) = y(\omega_1) + (\omega - \omega_1) \frac{dy(\omega)}{d\omega} \bigg|_{\omega_1} + \cdots \frac{j l}{x_L^2} (\omega - \omega_1) \frac{d^2 y}{d\omega^2} \bigg|_{\omega_1}
\]

For a rectangular waveguide,

\[
\frac{d\beta}{d\omega} = \frac{d}{d\omega} \sqrt{k_0^2 - k_c^2} = \frac{k_0}{\beta c}
\]

Thus

\[
y(\omega) = \frac{j \pi k_0 (\omega - \omega_1)}{\beta^2 c x_L^2}
\]

Use complex frequency,

\[
y(\omega) = \frac{\pi k_0 \omega_1}{2 \beta^2 c x_L^2} + \frac{j \pi k_0 (\omega - \omega_1)}{\beta^2 c x_L^2}
\]

This is similar to a parallel RLC circuit with

\[
R = \frac{2 \beta^2 c x_L^2}{\pi k_0 \omega_1} Z_0
\]

At critical coupling,

\[
X_L = Z_0 \sqrt{\frac{\pi k_0 \omega_1}{2 \beta^2 c}}
\]
Cavity Perturbations (6.8)

Material Perturbations

Let \((\mathbf{E}_0, \mathbf{H}_0)\) be the solution in a metallic cavity with material \((\varepsilon, \mu)\). Let \((\mathbf{E}, \mathbf{H})\) be the solution in the same cavity with material \((\varepsilon + \Delta \varepsilon, \mu + \Delta \mu)\). We have

\[
\nabla \times \mathbf{E}_0 = -j \omega \mu \mathbf{H}_0
\]

\[
\nabla \times \mathbf{H}_0 = j \omega \varepsilon \mathbf{E}_0
\]

\[
\nabla \times \mathbf{E} = -j \omega (\mu + \Delta \mu) \mathbf{H}
\]

\[
\nabla \times \mathbf{H} = j \omega (\varepsilon + \Delta \varepsilon) \mathbf{E}
\]

Then

\[
\nabla \cdot (\mathbf{E}_0^* \times \mathbf{H}) = \mathbf{H} \cdot \nabla \times \mathbf{E}_0^* - \mathbf{E}_0^* \cdot \nabla \times \mathbf{H} = j \omega \mu \mathbf{H} \cdot \mathbf{H}_0^* - j \omega (\varepsilon + \Delta \varepsilon) \mathbf{E}_0^* \cdot \mathbf{E}
\]

\[
\nabla \cdot (\mathbf{E} \times \mathbf{H}_0^*) = \mathbf{H}_0^* \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H}_0^* = j \omega \varepsilon \mathbf{E} \cdot \mathbf{E}_0^* - j \omega (\mu + \Delta \mu) \mathbf{H}_0^* \cdot \mathbf{H}_0^*
\]

By divergence theorem

\[
\int_{V_0} \nabla \cdot (\mathbf{E}_0^* \times \mathbf{H} + \mathbf{E} \times \mathbf{H}_0^*) \, dv = \oint_S (\mathbf{E}_0^* \times \mathbf{H} + \mathbf{E} \times \mathbf{H}_0^*) \cdot d\mathbf{s} = 0
\]

\[
= j \int_{V_0} \left[ \omega_0 \varepsilon - \omega (\varepsilon + \Delta \varepsilon) \right] \mathbf{E}_0^* \cdot \mathbf{E} + \left[ \omega_0 \mu - \omega (\mu + \Delta \mu) \right] \mathbf{H}_0^* \cdot \mathbf{H} \right) \, dv
\]

\[
\frac{\omega - \omega_0}{\omega} = \frac{\int_{V_0} (\Delta \varepsilon \mathbf{E} \cdot \mathbf{E}_0^* + \Delta \mu \mathbf{H} \cdot \mathbf{H}_0^*) \, dv}{\int_{V_0} (\varepsilon \mathbf{E} \cdot \mathbf{E}_0^* + \mu \mathbf{H} \cdot \mathbf{H}_0^*) \, dv}
\]

\[
\frac{\omega - \omega_0}{\omega} = \frac{\int_{V_0} (\Delta \varepsilon \mathbf{E} \cdot \mathbf{E}_0^* + \Delta \mu \mathbf{H} \cdot \mathbf{H}_0^*) \, dv}{\int_{V_0} (\varepsilon \mathbf{E} \cdot \mathbf{E}_0^* + \mu \mathbf{H} \cdot \mathbf{H}_0^*) \, dv}
\]

To sum up, as \(\varepsilon\) or \(\mu\) increases, the resonant frequency decreases.
Example 6.7

### Shape Perturbations

Let \((\vec{E}_0, \vec{H}_0)\) be the solution in a metallic cavity with material \((\varepsilon, \mu)\). Let \((\vec{E}, \vec{H})\) be the solution in the same cavity with shape perturbation \((\Delta V, \Delta S)\). We have

\[
\nabla \times \vec{E}_0 = -j \omega_0 \mu \vec{H}_0
\]
\[
\nabla \times \vec{H}_0 = j \omega_0 \varepsilon \vec{E}_0
\]
\[
\nabla \times \vec{E} = -j \omega(\mu) \vec{H}
\]
\[
\nabla \times \vec{H} = j \omega(\varepsilon) \vec{E}
\]

Then

\[
\nabla \cdot (\vec{E}_0^* \times \vec{H}) = \vec{H} \cdot \nabla \times \vec{E}_0^* - \vec{E}_0^* \cdot \nabla \times \vec{H} = j \omega_0 \mu \vec{H} \cdot \vec{H}_0^* - j \omega \vec{E}_0^* \times \vec{E}
\]
\[
\nabla \cdot (\vec{E} \times \vec{H}_0^*) = \vec{H}_0^* \cdot \nabla \times \vec{E} - \vec{E} \cdot \nabla \times \vec{H}_0^* = j \omega_0 \varepsilon \vec{E} \cdot \vec{E}_0^* - j \omega \mu \vec{H}_0^* \cdot \vec{H}
\]

By divergence theorem

\[
\int_{\nu} \nabla \cdot (\vec{E}_0^* \times \vec{H} + \vec{E} \times \vec{H}_0^*) dv = \oint_S (\vec{E}_0^* \times \vec{H} + \vec{E} \times \vec{H}_0^*) \cdot ds = \oint_S (\vec{E}_0^* \times \vec{H}) \cdot ds
\]

\[
= j \int_{\nu_0} \left[ \omega_0 \varepsilon - \omega \varepsilon \right] \vec{E}_0^* \cdot \vec{E} + \left[ \omega_0 \mu - \omega \mu \right] \vec{H}_0^* \cdot \vec{H} \right] dv
\]

Since \(S = S_0 - \Delta S\)

\[
\oint_S (\vec{E}_0^* \times \vec{H}) \cdot ds = \oint_{S_0} (\vec{E}_0^* \times \vec{H}) \cdot ds - \oint_{\Delta S} (\vec{E}_0^* \times \vec{H}) \cdot ds = -\oint_{\Delta S} (\vec{E}_0^* \times \vec{H}) \cdot ds
\]

\[
= -\oint_{\Delta S} (\vec{E}_0^* \times \vec{H}_0) \cdot ds = j \omega_0 \int_{\Delta V} (\varepsilon |\vec{E}_0|^2 - \mu |\vec{H}_0|^2) dv
\]

Thus
\[
\frac{\omega - \omega_0}{\omega_0} \int_{V_0} (-\varepsilon |\vec{E}_0|^2 + \mu |\vec{H}_0|^2) \, dv = \frac{\Delta W_m - \Delta W_e}{W_m + W_e}
\]