Aperture Antennas

Reflectors, horns.
High Gain
Nearly real input impedance

Huygens’ Principle

Each point of a wave front is a secondary source of spherical waves.

\[\text{Figure 7-1} \quad \text{Secondary waves used to construct successive wavefronts.}\]

\[\text{Figure 7-2} \quad \text{Plane wave incident on a slit in a screen. The edge diffraction leads to spreading of the radiation from the slit.}\]

\[1\text{In this chapter, the uppercase symbols } V \text{ and } S \text{ will be used to denote volume and surface.}\]
Equivalence Principle

Uniqueness Theorem: a solution satisfying Maxwell’s Equations and the boundary conditions is unique.

1. Original Problem (a): \((\vec{E}, \vec{H})\)
2. Equivalent Problem (b): \((\vec{E}, \vec{H})\) outside \(V\), \((\vec{E}_1, \vec{H}_1)\) inside \(V\), \((\vec{J}_{sl}, \vec{M}_{sl})\) on \(S\), where
   \[
   \vec{J}_{sl} = \hat{n} \times (\vec{H} - \vec{H}_1)
   \]
   \[
   \vec{M}_{sl} = -\hat{n} \times (\vec{E} - \vec{E}_1)
   \]
3. Equivalent Problem (c): \((\vec{E}, \vec{H})\) outside \(V\), zero fields inside \(V\), \((\vec{J}_s, \vec{M}_s)\) on \(S\), where
   \[
   \vec{J}_s = \hat{n} \times \vec{H}
   \]
   \[
   \vec{M}_s = -\hat{n} \times \vec{E}
   \]

To further simplify,
Case 1: PEC. No contribution from \(\vec{J}_s\).
Case 2: PMC. No contribution from \(\vec{M}_s\).

Infinite Planar Surface
To calculate the fields, first find the vector potential due to the equivalent electric and magnetic currents.

\[ \mathbf{A} = \mu \frac{e^{-j\beta r}}{4\pi r} \int \int_{S} \mathbf{j} \cdot (r')e^{j\beta r' r'} dS' \]

\[ \mathbf{F} = \varepsilon \frac{e^{-j\beta r}}{4\pi r} \int \int_{S} \mathbf{m} \cdot (r')e^{j\beta r' r'} dS' \]

In the far field, from Eqs. (1-105),

\[ \mathbf{E}_A \approx -j\omega (A_\theta \hat{\theta} + A_\phi \hat{\phi}) \]

\[ \mathbf{H}_F \approx -j\omega (F_\theta \hat{\theta} + F_\phi \hat{\phi}) \]

Since in the far field, the fields can be approximate by spherical TEM waves,
Thus the total electric field can be found by

\[ \vec{E} = \vec{E}_A + \vec{E}_F \approx -j \omega \eta \left( A_\theta \hat{\theta} + A_\phi \hat{\phi} \right) \]

Let \( (\vec{E}_a, \vec{H}_a) \) be the aperture fields, then

\[ \vec{A} = \mu \frac{e^{-jbr}}{4\pi r} \hat{n} \times \int \int_{S_a} \vec{H}_a e^{jB \cdot \vec{r}'} dS' \]

\[ \vec{F} = -\varepsilon \frac{e^{-jbr}}{4\pi r} \hat{n} \times \int \int_{S_a} \vec{E}_a e^{jB \cdot \vec{r}'} dS' \]

Let

\[ \vec{Q} = \int \int_{S_a} \vec{H}_a e^{jB \cdot \vec{r}'} dS' \]

\[ \vec{P} = \int \int_{S_a} \vec{E}_a e^{jB \cdot \vec{r}'} dS' \]

Use the coordinate system in Fig. 7-4, then

\[ P_x = \int \int_{S_a} \vec{E}_{ax}(x', y') e^{jB(x' \sin \theta \cos \phi + y' \sin \theta \sin \phi)} \, dx' \, dy' \]

\[ P_y = \int \int_{S_a} \vec{E}_{ay}(x', y') e^{jB(x' \sin \theta \cos \phi + y' \sin \theta \sin \phi)} \, dx' \, dy' \]

\[ Q_x = \int \int_{S_a} \vec{H}_{ax}(x', y') e^{jB(x' \sin \theta \cos \phi + y' \sin \theta \sin \phi)} \, dx' \, dy' \]

\[ Q_y = \int \int_{S_a} \vec{H}_{ay}(x', y') e^{jB(x' \sin \theta \cos \phi + y' \sin \theta \sin \phi)} \, dx' \, dy' \]

and
or in spherical coordinate system

\[
\mathbf{A} = \mu \frac{e^{-jbr}}{4\pi r} \left[ \hat{\theta} \cos(\theta) \left( \mathbf{Q}_x \sin\varphi - \mathbf{Q}_y \cos\varphi \right) + \hat{\phi} \left( \mathbf{Q}_x \cos\varphi + \mathbf{Q}_y \sin\varphi \right) \right]
\]

\[
\mathbf{F} = -\varepsilon \frac{e^{-jbr}}{4\pi r} \left[ \hat{\theta} \cos(\theta) \left( \mathbf{P}_x \sin\varphi - \mathbf{P}_y \cos\varphi \right) + \hat{\phi} \left( \mathbf{P}_x \cos\varphi + \mathbf{P}_y \sin\varphi \right) \right]
\]

Using Eq. (7-8), we have

\[
E_{\theta} = j\beta \frac{e^{-jbr}}{4\pi r} \left[ P_y \cos\varphi + P_x \sin\varphi \right] + \eta \cos(\theta) \left( \mathbf{Q}_y \cos\varphi - \mathbf{Q}_x \sin\varphi \right)
\]

\[
E_{\phi} = j\beta \frac{e^{-jbr}}{4\pi r} \left[ \cos(\theta) \left( \mathbf{P}_y \cos\varphi - \mathbf{P}_x \sin\varphi \right) \right] - \eta \left( \mathbf{Q}_y \sin\varphi + \mathbf{Q}_x \cos\varphi \right)
\]

If the aperture fields are TEM waves, then

\[
\mathbf{H}_a = \frac{1}{\eta} \hat{z} \times \mathbf{E}_a
\]

This implies

\[
E_{\theta} = j\beta \frac{e^{-jbr}}{2\pi r} \left[ \frac{1 + \cos(\theta)}{2} \right] \left( P_y \cos\varphi + P_x \sin\varphi \right)
\]

\[
E_{\phi} = j\beta \frac{e^{-jbr}}{2\pi r} \left[ \frac{1 + \cos(\theta)}{2} \right] \left( P_y \cos\varphi - P_x \sin\varphi \right)
\]

**Full Vector Form**

\[
\mathbf{E} = -j\omega \mathbf{A} - j\omega \eta \frac{\mathbf{R} \times \hat{F}}{4\pi r} - \int_{S_a} \oint \left[ \hat{n} \times \mathbf{E}_a - \eta \hat{n} \times (\hat{n} \times \mathbf{H}_a) \right] e^{j\beta \hat{r} \cdot \hat{s}} \, dS'
\]
The Uniform Rectangular Aperture

Let the electric field be

\[ \vec{E}_a = E_0 \hat{y}, \quad |x| \leq \frac{L_x}{2}, \quad |y| \leq \frac{L_y}{2} \]

Then,

\[ P_y = E_0 \int_{-L_x/2}^{L_x/2} e^{j\beta y'} \sin \theta \cos \phi \, dx' \int_{-L_y/2}^{L_y/2} e^{j\beta y} \sin \theta \cos \phi \, dy \]

\[ = E_0 L_x L_y \frac{\sin[(\beta L_x/2)u]}{(\beta L_x/2)u} \frac{\sin[(\beta L_y/2)v]}{(\beta L_y/2)v} \]

where \( u = \sin \theta \cos \phi, \ v = \sin \theta \sin \phi \)

Therefore,

\[ E_\theta = j\beta \frac{e^{-jBr}}{2\pi r} E_0 L_x L_y \frac{\sin[(\beta L_x/2)u]}{(\beta L_x/2)u} \frac{\sin[(\beta L_y/2)v]}{(\beta L_y/2)v} \]

\[ E_\phi = j\beta \frac{e^{-jBr}}{2\pi r} E_0 L_x L_y \cos \theta \cos \phi \frac{\sin[(\beta L_x/2)u]}{(\beta L_x/2)u} \frac{\sin[(\beta L_y/2)v]}{(\beta L_y/2)v} \]

At principle planes

\[ E_\theta = j\beta \frac{e^{-jBr}}{2\pi r} E_0 L_x L_y \frac{\sin[(\beta L_y/2)\sin \theta]}{(\beta L_y/2)\sin \theta}, \quad E_-\text{plane}(yz-\text{plane}) \]

\[ E_\phi = j\beta \frac{e^{-jBr}}{2\pi r} E_0 L_x L_y \cos \theta \frac{\sin[(\beta L_x/2)\sin \theta]}{(\beta L_x/2)\sin \theta}, \quad H_-\text{plane}(xz-\text{plane}) \]
For large aperture \((L_x, L_y > \lambda)\), the main beam is narrow, the \(\cos \theta\) factor is negligible. The half-power beam width

\[
HP_x = 50.8 \frac{\lambda}{L_x} \text{ deg}
\]

\[
HP_y = 50.8 \frac{\lambda}{L_y} \text{ deg}
\]

Also,

\[
\Omega_A = \int_0^{2\pi/2} \int_0^{\pi/2} |F(\theta, \phi)|^2 d\Omega = \frac{\lambda^2}{L_x L_y} \Rightarrow D_u = \frac{4\pi}{\Omega_A} = \frac{4\pi}{\lambda^2} L_x L_y = \frac{4\pi}{\lambda^2} A_p
\]

\[
\therefore A_{em} = A_p
\]

Example: a \(20\lambda \times 10\lambda\) Uniform Rectangular Aperture

\[\text{Figure 7-7} \quad \text{Contour plot of the pattern from a uniform amplitude, uniform phase rectangular aperture \((L_x = 20\lambda, L_y = 10\lambda)\). The solid contour levels are 0, \(-5, -10, -15, -20, -25, -30\) dB. The dashed contour levels are \(-35\) and \(-40\) dB. Principal plane profiles are shown in Fig. 7-8.}\]
Figure 7.8 Principal plane patterns for a uniform amplitude, uniform phase rectangular aperture ($L_x = 20\lambda$, $L_y = 10\lambda$). The complete pattern is shown in Fig. 7.7.
Figure 7-9 Geometry for an open-ended rectangular waveguide operating in the dominant TE_{m} mode as in Example 7-3. The aperture electric field \( \mathbf{E}_a \) and radiated field components \( E_x \) and \( E_y \) are shown.

Figure 7-10 E-plane radiation patterns of the open-ended WR-90 waveguide of Example 7-3 operating at 9.32 GHz with the geometry of Fig. 7-9. Patterns are calculated using (7-58b) for free space (solid curve) and (7-57b) for a ground plane (dashed curve). Also shown is the measured pattern (dotted curve) for the open-ended waveguide in free space [4].
Technical for Evaluating Gain

Directivity

From (7-27), (7-24), (7-61)

\[ U(\theta,\varphi) = \frac{\beta^2}{32\pi^2 \eta} (1 + \cos\theta)^2 \left[ |P_x|^2 + |P_y|^2 \right] \]

Thus, for broadside case,

\[ U_m = \frac{\beta^2}{8\pi^2 \eta} \left| \int \int_{S_a} \vec{E}_{a} \ dS' \right|^2 \]

Total power

\[ P = \frac{1}{2\eta} \int \int_{S_a} |\vec{E}_{a}|^2 \ dS' \]

Then,

\[ D = \frac{4\pi U_m}{P} = \frac{4\pi}{\lambda^2} \left( \int \int_{S_a} |\vec{E}_{a}|^2 \ dS' \right)^2 \]

In general, for uniform distribution

\[ D_u = \frac{4\pi}{\lambda^2} A_p \]

If

\[ E_a(x,y) = E_{a_1}(x)E_{a_2}(y) \]

then

\[ D = \pi D_x D_y \cos\theta_0 \]

where \( D_x, D_y \) are the directivity of a line source due to \( E_{a_1}(x), E_{a_2}(y) \) respectively. \( \theta_0 \) the main beam direction relative to broadside.
Directivity of an Open-Ended Rectangular Waveguide:

\[ D = \frac{4\pi}{\lambda^2} (0.81) ab \]

**Gain and Efficiencies**

\[ G = \frac{4\pi}{\lambda^2} A_e = \frac{4\pi}{\lambda^2} \varepsilon_{ap} A_p = \varepsilon_{ap} D_u \]

where \( \varepsilon_{ap} = \varepsilon_r \varepsilon_t \varepsilon_s \varepsilon_a \)

\( \varepsilon_{ap} \): aperture efficiency
\( \varepsilon_r \): radiation efficiency. (~1 for aperture antennas)
\( \varepsilon_t = \frac{D_t}{D_u} \): taper efficiency or utilization factor.
\( \varepsilon_s \): spillover efficiency. \( \varepsilon_s \varepsilon_t \) is called \( \varepsilon_i \): illumination efficiency.
\( \varepsilon_a = \varepsilon_{cr} \varepsilon_{ph} \cdots \): achievement efficiency. \( \varepsilon_{cr} \): cross-polarization efficiency. \( \varepsilon_{ph} \): phase-error efficiency.

**Beam efficiency**

\[ \varepsilon_M = \frac{\int \int |F(\theta, \phi)|^2 d\Omega}{\Omega_M} = \frac{\int \int |F(\theta, \phi)|^2 d\Omega}{\Omega_A} \]

**Simple Directivity Formulas in Terms of HP beam width**

1. Low directivity, no sidelobe

\[ D = \frac{4\pi}{HP_E \cdot HP_H} = \frac{41253}{HP_E^* \cdot HP_H^*} \]

2. Large electrical size
3. High gain

\[ D = \frac{26000}{\text{HP}_E \text{HP}_H} \]

Example 7-5: Pyramidal Horn Antenna (aperture efficiency=0.51)
Measured gained at 40 GHz: 24.7 dB.
A=5.54 cm, B=4.55 cm.
\[ \text{HP}_E^\circ = 9^\circ, \text{HP}_H^\circ = 10^\circ \]
\[ G = 0.51 \frac{4\pi}{\lambda^2} AB = 24.6 \text{ dB} \]
\[ D = \frac{26000}{\text{HP}_E \text{HP}_H} = 24.6 \text{ dB} \]

Example 7-6: Circular Parabolic Reflector Antenna
Typical aperture efficiency: 55%.
Diameter: 3.66 m
Frequency: 11.7 GHz
Measured Gain: 50.4 dB.
Measured \[ \text{HP}_E^\circ = \text{HP}_H^\circ = 0.5^\circ \].

1. Computed by aperture efficiency
\[ G = \varepsilon_{ap} \frac{4\pi}{\lambda^2} A_p = 50.4 \text{ dB} \]

2. Computed by half power beam width
\[ G = \frac{26000}{\text{HP}_E \text{HP}_H} = 50.2 \text{ dB} \]
**Rectangular Horn Antenna**

(a) H-plane sectoral horn.  \( E_z \)  

(b) E-plane sectoral horn.  \( E_z \) 

(c) Pyramidal horn.  \( E_z \)  

*Figure 7-11* Rectangular horn antennas.

High gain, wide band width, low VSWR

(a) Overall geometry.  \( E_z \)  

(b) Cross section through the az-plane (H-plane).  \( E_z \)  

*Figure 7-12* H-plane sectoral horn antenna.

**H-Plane Sectoral Horn Antenna**

Evaluating phase error \( e^{-j\beta(R-R_1)} \)

\[
R = \sqrt{R_1^2 + x^2} \approx R_1 \left[ 1 + \frac{1}{2} \left( \frac{x}{R_1} \right)^2 \right]
\]

thus the aperture electric field distribution
where $I(\theta, \phi)$ is defined in (7-108), (7-109)

Directivity

$$D_H = \frac{4\pi}{\lambda^2} \epsilon_t \epsilon_{ph}^H Ab$$

Figure 7-13: universal E-plane and H-plane pattern with \( \frac{1 + \cos \theta}{2} \)

factor omitted, and \( t = \frac{1}{8} \left( \frac{A}{\lambda} \right)^2 \frac{1}{R_1/\lambda} \) (a measure of the maximum phase error at the edge)

Figure 7-14: Universal directivity curves.

Optimum directivity occurs at \( A = \sqrt{3} \lambda R_1 \) and \( t_{op} = \frac{A^2}{8\lambda R_1} = \frac{3}{8} \)

From figure 7.13 for optimum case,

$$HP_{H} \approx 1.36 \frac{\lambda}{A} = 78^\circ \frac{\lambda}{A}$$
Figure 7-13 Universal radiation patterns for the principal planes of an H-plane sectoral horn as shown in Fig. 7-12. The factor $(1 + \cos \theta)^2$ is not included.

Figure 7-14 Universal directivity curves for an H-plane sectoral horn.
E-Plane Sectoral Horn Antenna

The aperture electric field distribution

\[ E_{ay} = E_0 \cos \frac{\pi x}{A} e^{-j \left( \frac{B}{2R_2} \right) y^2} \]

See (7-129) for the resulting \( \vec{E} \)

Directivity

\[ D_E = \frac{4\pi}{\lambda^2} \varepsilon_i \varepsilon_{ph} \alpha B \]

Figure 7-16: universal E-plane and H-plane pattern with \( \frac{1 + \cos \theta}{2} \) factor omitted, and \( s = \frac{1}{8} \left( \frac{B}{\lambda} \right)^2 \frac{1}{R_2/\lambda} \) (a measure of the maximum phase error at the edge)

Figure 7-17: Universal directivity curves.

Optimum directivity occurs at \( B = \sqrt{2\lambda R_2} \) and \( s_{op} = \frac{1}{4} \)

From figure 7.13,
$$HP_E \approx 0.94 \frac{\lambda}{B} = 54^\circ \frac{\lambda}{B}$$

Figure 7.16 Universal radiation patterns for the principal planes of an E-plane.

Figure 7.17 Universal directivity curves for an E-plane sectoral horn.
Pyramidal Horn Antenna

![Diagram of a pyramidal horn antenna with labels and geometric details](image)

The aperture electric field distribution

\[ E_{ay} = E_0 \cos \frac{\pi x}{A} e^{-j \left( \frac{\sqrt{b}}{2} \frac{x^2 + y^2}{R_1 + R_2} \right)} \]

\[ D_p = \frac{\pi}{32} \left( \frac{\lambda}{a} D_E \left( \frac{\lambda}{b} D_H \right) \right) \]

At optimum condition:

\[ \varepsilon_{ap} = \varepsilon_r \varepsilon_{ph} \left( s = \frac{1}{4} \right) \varepsilon_{ph} \left( t = \frac{3}{8} \right) = 0.81 \times 0.8 \times 0.79 = 0.51 \]

Optimum gain \( G = 0.51 \frac{4\pi A B}{\lambda^2} \)

For non optimum case,

\[ \varepsilon_{ph}^E(s) = 1.00329 - 0.11911s - 2.75224s^2, \ 0 \leq s \leq 0.262. \]

\[ \varepsilon_{ph}^H(t) = 1.00323 - 0.08784t - 1.27048t^2, \ 0 \leq t \leq 0.397. \]

Design procedure:

1. Specify gain \( G \), wavelength \( \lambda \), waveguide dimension \( a, b \).
2. Using $\varepsilon_{ap} = 0.51$, determine $A$ from the following equation

$$A^4 - aA^3 + \frac{3bG\lambda^2}{8\pi\varepsilon_{ap}} A = \frac{3G^2\lambda^4}{32\pi^2\varepsilon_{ap}^2}$$

3. Determine $B$ from $G = 0.51 \frac{4\pi}{\lambda^2} AB$

4. Determine $R_1$, $R_2$ by $A = \sqrt{3\lambda R_1}$, $B = \sqrt{2\lambda R_2}$

5. Determine $R_H$, $R_E$ by $\frac{R_1}{R_H} = \frac{A}{A-a}$, $\frac{R_2}{R_E} = \frac{B}{B-b}$

6. Determine $l_H$, $l_E$ by $l_H^2 = R_1^2 + \left(\frac{A}{2}\right)^2$, $l_E^2 = R_2^2 + \left(\frac{B}{2}\right)^2$

7. Verify if $R_E = R_H$ and $s = 0.25$, $t = 0.375$ by

$$t = \frac{A^2}{8\lambda R_1}, \quad s = \frac{B^2}{8\lambda R_2}$$

Example 7-7: Design a X-band (8.2 to 12.4 GHz) standard horn fed by WR90 (0.9 in × 0.6 in) waveguide.

**Goal:** $G = 21.75$ dB at 8.75 GHz.

1. Solve for $A$: $A = 18.61 \text{ cm}$

2. Solve for the rest parameters:

   $B = 14.75 \text{ cm}$, $R_1 = 33.67 \text{ cm}$, $R_2 = 31.72 \text{ cm}$

   $l_H = 34.93 \text{ cm}$, $l_E = 32.56 \text{ cm}$, $R_H = R_E = 29.53 \text{ cm}$

3. Evaluating the gain by Fig. 7-14 and 7-17:

   $$G = \pi \frac{\lambda}{32} \left( \frac{D_E}{a} \right) \left( \frac{\lambda}{b} \frac{D_H}{b} \right) = \pi \frac{36 \times 43}{32} = 152 = 21.8 \text{ dB}$$

   $$G = \varepsilon_{ap} \frac{4\pi}{\lambda^2} AB = 21.79 \text{ dB} \text{ (exact phase error)}$$
\[ G = \frac{26000 \text{ } HP_E^\circ}{HP_H^\circ} = \frac{26000}{12.4^\circ 14.2^\circ} = 21.7 \text{ dB} \]
Figure 7-19 Aperture efficiencies for E- and H-plane sectoral horns (left ordinate) and phase efficiencies associated with E- and H-plane flares (right ordinate).

Figure 7-20 Directivity and aperture efficiency of the standard gain rectangular horn of Example 7-7.

Figure 7-21 Principal plane patterns for the optimum pyramidal horn antenna of Example 7-7 at 8.75 GHz. The patterns include the 
\((1 + \cos \theta)/2\) factor. HP_E = 12.4° and HP_H = 14.2°.
Reflector Antennas

Parabolic Reflector

Parabolic equation:

\[ \rho^2 = 4F(F - z_f), \quad \rho \leq \frac{D}{2} \]

or

\[ r_f = \frac{2F}{1 + \cos \theta_f} = F \sec^2 \frac{\theta_f}{2} \]

Properties

1. Focal point at \( O \). All rays leaving \( O \), will be parallel after reflection from the parabolic surface.
2. All path lengths from the focal point to any aperture plane are equal.
3. To determine the radiation pattern, find the field distribution at the aperture plane using GO.

Geometrical Optics (GO)
Requirements
1. The radius curvature of the reflector is large compared to a wavelength, allowing planar approximation.
2. The radius curvature of the incoming wave from the feed is large, allowing planar approximation.
3. The reflector is a perfect conductor, thus the reflect coefficient $\Gamma = -1$.

Parabolic reflector:
Wideband.
Lower limit determined by the size of the reflector. Should be several wavelengths for GO to hold.
Higher limit determined by the surface roughness of the reflector. Should much smaller than a wavelength.
Also limited by the bandwidth of the feed.

Determining the power density distribution at the aperture by

$$\vec{E}_a(\theta', \phi') = V_0 e^{-j\beta_2 r_f} \frac{F_f(\theta_f, \phi_f)}{r_f} \hat{u}_r$$

$$\hat{u}_r = 2(\hat{n} \times \hat{u}_i) \hat{n} - \hat{u}_i$$

where $\hat{u}_r = \frac{\vec{E}_r}{|\vec{E}_r|}$, $\hat{u}_i = \frac{\vec{E}_i}{|\vec{E}_i|}$

$$\vec{P} = V_0 \int_{0}^{2\pi} \int_{0}^{\pi} \frac{F_f(\theta, \phi)}{r_f} \hat{u}_r e^{-j\beta_0 \sin \theta \cos (\phi - \phi')} \rho^' d\rho^' d\phi^'$$

PO/surface current method

$$\vec{E} = -j\omega \mu e^{-j\beta r} \int_{S_r} \left[ \vec{J}_s - (\vec{J}_s \cdot \hat{r}) \hat{r} \right] e^{j\beta r^'} dS^'$$
\[ \mathbf{J}_S = 2\hat{n} \times \mathbf{H}_i \]

PO and GO both yield good patterns in main beam and first few sidelobes. Deteriorate due to diffraction by the edge of the reflector. PO is better than GO for offset reflectors.

**Axis-symmetric Parabolic Reflector Antenna**

For a linear polarized feed along x-axis, the pattern can be approximate by the two principle plan patterns as below.

\[ \mathbf{E}_a = -j\omega \mu \frac{e^{-j\beta zF}}{4\pi r_f} \left\{ -\hat{x}[C_E(\theta)\cos^2\phi_f + C_H(\theta)\sin^2\phi_f] + \hat{y}[C_E(\theta) - C_H(\theta)]\sin\phi_f\cos\phi_f \right\} \]

where \( C_E, C_H \) are E-plane and H-plane patterns.

If the pattern is rotationally symmetric, then \( C_E = C_H \). We have

\[ \mathbf{E}_a \approx -j\omega \mu \frac{e^{-j\beta zF}}{4\pi r_f} \left\{ \hat{x}[F(\theta)] \right\} \]

Also, the cross-polarization of the aperture field is maximum in the \( \phi_f = 45^\circ, 135^\circ \). \( C_E \neq C_H \) Leads to cross-polarization.

For a short dipole, \( C_E = \cos\phi_f, C_H = 1 \),

\[ \mathbf{E}_a \approx -j\omega \mu \frac{e^{-j\beta zF}}{4\pi r_f} \left\{ -\hat{x}[\cos\phi_f\cos^2\phi_f + \sin^2\phi_f] + \hat{y}[\cos\phi_f - 1]\sin\phi_f\cos\phi_f \right\} \]

At \( \phi_f = 0^\circ, 90^\circ \), only x component exists.
Summary:
1. F/D increases, cross-polarization decreases. Since the range of $\theta_f$ decreases as F/D increases, the term $1-\cos\theta_f \approx 0$.
2. Fields inverted because of reflection from conductor.
3. Cross-polarization cancels each other on principal plane in the far field.
Figure 7.29 An axisymmetric parabolic reflector with diameter $D = 100\lambda$ and $F/D = 0.5$ fed by a half-wave dipole located at the focus. All data were computed using GRASP [29]. From [43] © 1993. Reprinted by permission of Artech House, Inc., Boston, MA.
(c) Cross-polarization contours (normalized)

(d) Co- and cross-polarization patterns in $\phi = 45^\circ, 135^\circ$ plane.
The peak cross polarization is 26.3 dB below the co-polar peak.

Figure 7-29 (continued)
Approximation formula

Normalized aperture field

\[ E_{an}(\rho') = F_j(\theta_j = 2\tan^{-1}\frac{\rho'}{2F}) \cdot \frac{1}{1 + (\frac{\rho'}{2F})^2} = 20\log|F_j| - 20\log \left[ 1 + \left(\frac{\rho'}{2F}\right)^2 \right] \text{(dB)} \]

Thus,

\[ EI = 20\log\left[ E_{an}(\rho' = a) \right] = -FT - L_{\text{sph}} = -ET \text{ (dB)} \quad (7-208) \]

where

- \( EI \) = edge illumination (dB) = 20 log C
- \( ET \) = edge taper (dB) = \(-EI\)
- \( FT \) = feed taper (at aperture edge) (dB) = \(-20\log[F_j(\theta_0)]\)

Spherical spreading loss at the aperture edge

\[ L_{\text{sph}} = 20\log \left[ 1 + \frac{1}{16\left(\frac{F}{D}\right)^2} \right] = -20\log \left[ \frac{1 + \cos\theta_0}{2} \right] \]

Design procedure of axial symmetrical aperture field:

1. Estimate \( EI \) by the radiation pattern of the feed at the edge angle \( \theta_0 \) of the reflector.
2. Calculate \( L_{\text{sph}} \) due to the distance from the feed to the edge.
3. Estimate \( ET \) at the aperture by adding the \( EI \) and \( L_{\text{sph}} \).
4. Look up Table 7-1 for a suitable n.

Example 7-8: A 28-GHz Parabolic Reflector Antenna fed by circular corrugated horn.

\( f = 28.56 \text{ GHz, } D = 1.22 \text{ m, } F/D = 0.5, HP = 56^\circ, BW_{-10\text{dB}} = 104^\circ. \)

\( \theta_0 = 53.1^\circ > BW_{-10\text{dB}}/2. \)
Assume $FT=11 \text{ dB}$, then $EI=-12.9 \text{ dB}=0.2265$

Use Table 7.1b for $n=2$ and interpolate, we have

$$HP = 1.214 \frac{\lambda}{D} = 0.599^\circ \ (0.605^\circ \text{ measured})$$

$$SLL = -30 \text{ dB} \ (-28.5 \text{ dB \ measured})$$

*Figure 7-30* Measured (solid) $E$-plane pattern for the 1.22-m-diameter axisymmetric parabolic reflector at 28.56 GHz in Example 7-8. The computed (dashed) pattern is for a parabolic-squared circular aperture distribution on a $-12.9$-dB pedestal.
Offset Parabolic Reflectors

Reduce blocking loss.
Increase cross-polarization.

Dual Reflector Antenna

Spill over energy directed to the sky.
Compact.
Simplify feeding structure.
Allow more design freedom. Dual shaping.
Other types

Design example

1. Determine the reflector diameter by half-power beam width. For the optimum -11 dB edge illumination,

\[ HP = 1.18 \frac{\lambda}{D} \text{ rad} \] (7-248)

2. Choose F/D. Usually between 0.3 to 1.0.

3. Determine the required feed pattern using \( \cos^q \theta_f \) model.

\[ q = \frac{\log \left[ EI \left( 1 + \frac{1}{16(F/D)^2} \right) \right]}{\log \left[ \cos \left( 2\tan^{-1} \frac{1}{4(F/D)} \right) \right]} \] (7-249)

Example 7-9: \( f=1 \text{ GHz} \), \( HP=1^\circ \)

From 7-248
\[
D = \frac{1.18\lambda}{\text{HP} \frac{\pi}{180^\circ}} = 2.0 \text{ m}
\]

Choose F/D=0.5, then \(\theta_0 = 2\tan^{-1}\frac{1}{4F/D} = 53.1^\circ\).

From (7.249), find \(q\) to approximate the pattern by \(\cos^q(\theta_f)\).

\[ q = 2.0551 \approx 2 \]

Verify \(EI = -11 \text{ dB}\), by (7-208).

\[ EI = -FT - L_{sp} = -8.86 - 1.93 = -10.79 \text{ (dB)} \]

\(\varepsilon_i = 0.82\)

\(\varepsilon_s = 0.92\)

\(\varepsilon_t = \frac{\varepsilon_i}{\varepsilon_s} = 0.89\)

Note: for feed pattern \(F_f(\theta_f) = \begin{cases} \cos^q\theta_f & \theta_f \leq \frac{\pi}{2} \\ 0 & \theta_f > \frac{\pi}{2} \end{cases}\)

Consideration of Gain due to taper and spillover:

1. The more taper, the less loss due to spillover, but less taper efficiency \(\varepsilon_t\).
2. The less taper, the more loss due to spillover, but higher taper efficiency \(\varepsilon_t\).
3. Optimum efficiency for \(\cos^2\theta_f\) feed: \(EI = -11 \text{ dB}\).
### Table 7-1 Characteristics of Tapered Circular Aperture Distributions

**a. Parabolic taper**

\[
E_d(p') = \left[ 1 - \left( \frac{p'}{a} \right)^2 \right]^n
\]

\[
f(\theta, n) = \frac{2^{n+1}(n+1)!J_{n+1}(\beta a \sin \theta)}{(\beta a \sin \theta)^{n+1}}
\]

<table>
<thead>
<tr>
<th>(n)</th>
<th>HP (rad)</th>
<th>Side Lobe Level (dB)</th>
<th>(e_i)</th>
<th>Normalized Pattern (f(\theta, n))</th>
<th>Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(1.02 \frac{A}{2a})</td>
<td>-17.6</td>
<td>1.00</td>
<td>(\frac{2J_1(\beta a \sin \theta)}{\beta a \sin \theta})</td>
<td>Uniform</td>
</tr>
<tr>
<td>1</td>
<td>(1.27 \frac{A}{2a})</td>
<td>-24.6</td>
<td>0.75</td>
<td>(8J_2(\beta a \sin \theta))</td>
<td>Parabolic</td>
</tr>
<tr>
<td>2</td>
<td>(1.47 \frac{A}{2a})</td>
<td>-30.6</td>
<td>0.55</td>
<td>(48J_3(\beta a \sin \theta)</td>
<td>Parabolic squared</td>
</tr>
</tbody>
</table>

**b. Parabolic taper on a pedestal**

\[
E_d(p') = C + (1 - C)\left[ 1 - \left( \frac{p'}{a} \right)^2 \right]^n
\]

\[
f(\theta, n, C) = \frac{C f(\theta, n = 0) + \frac{1 - C}{n + 1} f(\theta, n)}{C + \frac{1 - C}{n + 1}}
\]

<table>
<thead>
<tr>
<th>Edge Illumination</th>
<th>(n = 1)</th>
<th>(n = 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C_{de})</td>
<td>HP (rad)</td>
<td>Side Lobe Level (dB)</td>
</tr>
<tr>
<td>-8</td>
<td>0.398</td>
<td>1.12 (\frac{A}{2a})</td>
</tr>
<tr>
<td>-10</td>
<td>0.316</td>
<td>1.14 (\frac{A}{2a})</td>
</tr>
<tr>
<td>-12</td>
<td>0.251</td>
<td>1.16 (\frac{A}{2a})</td>
</tr>
<tr>
<td>-14</td>
<td>0.200</td>
<td>1.17 (\frac{A}{2a})</td>
</tr>
<tr>
<td>-16</td>
<td>0.158</td>
<td>1.19 (\frac{A}{2a})</td>
</tr>
<tr>
<td>-18</td>
<td>0.126</td>
<td>1.20 (\frac{A}{2a})</td>
</tr>
<tr>
<td>-20</td>
<td>0.100</td>
<td>1.21 (\frac{A}{2a})</td>
</tr>
</tbody>
</table>
Table 7-1 (continued)

b. Parabolic taper on a Pedestal (continued)

Interpolation equations for finding HP and $e_t$ when $C_{db}$ is between $-8$ and $-20$ dB:

<table>
<thead>
<tr>
<th>Quantity</th>
<th>$n = 1$</th>
<th>$n = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>HP</td>
<td>$k = -0.008C_{db} + 1.06$</td>
<td>$k = -0.015C_{db} + 1.02$</td>
</tr>
<tr>
<td>$e_t$</td>
<td>$e_t = 0.01C_{db} + 1.02$</td>
<td>$e_t = 0.019C_{db} + 1.06$</td>
</tr>
</tbody>
</table>

Figure 7-39 Aperture taper $e_n$, spillover $e_s$, and illumination $e_t$ efficiencies for a $\cos^2 \theta$ feed pattern ($q = 2$) as a function of edge illumination $E_l$.

Figure 7-38 Illustration of the influence of the feed antenna pattern on reflector aperture taper and spillover.

(a) Broad feed pattern giving high aperture taper efficiency but low spillover efficiency.

(b) Narrow feed pattern giving high spillover efficiency but low aperture taper efficiency.
Figure 7.43 Aperture field distribution for axisymmetric parabolic reflector of Example 7.9 (dashed curve) along with the parabolic-squared-on-a-pedestal distribution with $C = 0.28$ (solid curve).

Figure 7.44 Pattern for the 2-m axisymmetric parabolic reflector antenna of Example 7.9 computed using the PRAC code.
A General Approximate Feed Model for Broad Main Beam and Peak at $\theta_f=0$

Assume
\[ C_E(\theta_f) = \cos^{q_E} \theta_f, \quad C_H(\theta_f) = \cos^{q_H} \theta_f, \quad \theta_f < \frac{\pi}{2} \]

That is the feed pattern $F_f(\theta_f) = \begin{cases} \cos^{q_f} \theta_f, & \theta_f < \frac{\pi}{2} \\ 0, & \theta_f > \frac{\pi}{2} \end{cases}$

For symmetrical feed,
\[ C_E(\theta_f) = C_H(\theta_f) = \cos^q \theta_f, \quad \theta_f < \frac{\pi}{2} \]

The value of $q$ is chosen to match the real feed pattern at $\theta_f'$ (usually $\theta_0$) by
\[ q = \frac{\log \left| F_f(\theta_f') \right|}{\log \left| \cos \theta_f' \right|} \]

Then,
\[ G_f = \frac{2(2q_E+1)(2q_H+1)}{q_E+q_H+1} \]

For symmetrical feed, the illumination efficiency $\varepsilon_i$, the spill over efficiency $\varepsilon_s$, the feed gain $G_f$ and $EI$ are computed by
\[ G_f = 2(2q+1) \]
\[ \varepsilon_s = 1 - \cos^{2q+1} \theta_0 \]
\[ \varepsilon_i = \cot^2 \frac{\theta_0}{2} \begin{cases} 
24 \left[ \sin^2 \frac{\theta_0}{2} + \ln \left( \cos \frac{\theta_0}{2} \right) \right]^2 & q = 1 \\
40 \left[ \sin^4 \frac{\theta_0}{2} + \ln \left( \cos \frac{\theta_0}{2} \right) \right]^2 & q = 2 \\
14 \left[ \frac{1}{2} \sin^2 \theta_0 + \frac{1}{3} (1 - \cos \theta_0)^3 + 2 \ln \left( \cos \frac{\theta_0}{2} \right) \right]^2 & q = 3 
\end{cases} \]

\[ EI = \frac{1 + \cos \theta_0}{2} \cos^q \theta_0 \]