Linear Systems

- Textbook: Strum, “Contemporary Linear Systems using MATLAB.”

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Chap. 1 Basic Concepts

Continuos V. S. Discrete

![Graph showing analog and sampled signals](image)

**FIGURE 1.3** Analog signal and sequence resulting from the sampling process

Continuous System

![Diagram of continuous system](image)

**FIGURE 1.2** Continuous system

Discrete System

![Diagram of discrete system](image)

**FIGURE 1.4** Discrete system
Unit Impulse

1. A virtual function.
2. Any arbitrary analog signal can be decomposed by impulses.
3. The response of any analog signal can be decomposed by impulse responses.

Properties:
\[ \delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases} \]
\[ \int_{-\infty}^{0} \delta(t) dt = 1 \]
\[ \int_{-\infty}^{\infty} f(t) \delta(t-t_1) dt = f(t_1) \] (Sifting Property)
\[ f(t) = \int_{-\infty}^{\infty} f(\tau) \delta(t-\tau) d\tau \]

Example:
\[ \delta(t) = \lim_{\Delta \to 0} \left\{ \begin{array}{ll} \frac{1}{\Delta}, & 0 \leq t \leq \Delta \\ 0, & \text{otherwise} \end{array} \right. \]
Unit Step Function

\[ u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases} \]

Representing Signals

a. \[ f(t) = \frac{A}{2} [u(t) - u(t-2)] \]

b. \[ f(t) = \sum_{m=-\infty}^{\infty} \frac{A}{2} (t-2m) [u(t-2m) - u(t-2m-2)] \]
Conversion between Continuous and Discrete Signals

Sampling Theorem: Let

\( f_{\text{max}} \): the maximum frequency component of the signal,
\( f_s \): the sampling frequency.

then \( f_s \geq 2f_{\text{max}} \) if the signal can be uniquely represented by the samples, or the signal can fully recovered from the samples.
Chap. 2 Continuous Systems

Basic Concept

**Linearity**: A system is linear if and only if it satisfies the principle of homogeneity and additivity.
1. Homogeneity
2. Additivity
3. Homogeneity and additivity

**a. Principle of homogeneity**

\[ x_1(t) \rightarrow \text{Linear system} \rightarrow y_1(t) \]

\[ x(t) = C_1 x_1(t) \rightarrow \text{Linear system} \rightarrow y(t) = C_1 y_1(t) \]

**b. Principle of additivity**

\[ x_1(t) \rightarrow \text{Linear system} \rightarrow y_1(t) \]

\[ x(t) = x_1(t) + x_2(t) \rightarrow \text{Linear system} \rightarrow y(t) = y_1(t) + y_2(t) \]

\[ x_2(t) \rightarrow \text{Linear system} \rightarrow y_2(t) \]

**c. Principle of superposition**

\[ x_1(t) \rightarrow \text{Linear system} \rightarrow y_1(t) \]

\[ x(t) = C_1 x_1(t) + C_2 x_2(t) \rightarrow \text{Linear system} \rightarrow y(t) = C_1 y_1(t) + C_2 y_2(t) \]

**FIGURE 2.3 Linear**
**Time Invariance:** The same input applied at different times will produce the same output except shifted in time. That is, for arbitrary $t_0$

![Diagram](image)

**Linear Time-Invariant Systems (LTI):** Linear + time-invariant.
1. Can be analyzed by Laplace and Fourier transforms.
2. In time domain, described by linear differential equations with constant coefficients.
3. In transform domains, described by linear algebraic equation.

**Causality:** Output depends on only previous inputs. That is, $y(t)$ only depends on $x(\tau)$, $\tau \leq t$.

![Graphs](image)
Stability: if the input is bounded, the output is also bounded (bounded-input-bounded-output BIBO).

Problems 2.1. Discuss the properties of the following continuous systems.
   a. \( y(t) = Kx(t) + A \) (i) Linear? (ii) Time invariant?
   b. \( y(t) = \int_{0}^{t} x(\tau) d\tau + y(0), \quad t \geq 0 \). Causal or noncausal?
   c. \( y(t) = t|x(t)|^2 \)

Nth-Order Differential Equation Model
In general, a single-input-single-output LTI system can be modeled by Nth-order differential equation as follows,
\[
a_0 y(t) + a_1 \frac{dy(t)}{dt} + \ldots + a_N \frac{d^N y(t)}{dt^N} = b_0 x(t) + b_1 \frac{dx(t)}{dt} + \ldots + b_N \frac{d^N x(t)}{dt^N}
\]

Example:

![Circuit Diagram]

a. Analog filter

\[
\frac{d^2 v(t)}{dt^2} + \frac{2}{RC_1} \frac{dv(t)}{dt} + \frac{1}{R^2 C_1 C_2} v(t) = \frac{1}{R^2 C_1 C_2} e(t)
\]

Initial Condition Solution of a Differential Equation
1. Find the homogeneous solution, i.e.
Let $y(t) = Ce^{st}$, substitute this to the above equation. Then,

$$a_0 s^0 Ce^{st} + a_1 s^1 Ce^{st} + \ldots + a_N s^N Ce^{st} = 0 \Rightarrow a_0 s^0 + a_1 s^1 + \ldots + a_N s^N = 0$$

which is a polynomial called characteristic equation.

Let $r_1, r_2, \ldots, r_N$ be the roots of the characteristic equation, then

$$\sum_{k=0}^{N} a_k s^k = a_N (s-r_1)(s-r_2) \ldots (s-r_N) = 0$$

Note: if stable, the real parts of the roots must be negative.

The initial condition solution is in the form

$$y_{IC} = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \ldots + C_N e^{r_N t}$$

where $C_1, C_2, \ldots, C_N$ are unknown coefficients to be determined by the initial conditions

$$y(0), \frac{dy(0)}{dt}, \ldots, \frac{d^{N-1}y(0)}{dt^{N-1}}.$$

**Problem 2.2**

Let

$$\ddot{\theta}(t) + 3 \dot{\theta}(t) + 2 \theta(t) = \frac{1}{3} \lambda(t)$$

a. Find the system’s characteristic equation.
b. Find the root.
c. Is the system stable?
d. Find the general form of the homogeneous solution.
e. Find the initial solution for $\theta(0) = 1$ and $\dot{\theta}(0) = 0$.

**The Unit Impulse Response Model**

Let the input be $\delta(t)$, and output $h(t)$, then $h(t)$ is called impulse response. For any input $x(t)$, we have
\[ y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) \, d\tau \]

**Sifting property:**
\[ \int_{-\infty}^{\infty} f(t) \delta(t-t_0) \, dt = f(t_0) \]

**Stability Definition:**
\[ \int_{-\infty}^{\infty} |h(\tau)| \, d\tau < \infty \]

**Problem 2.3**
\[ y(t) = \int_{0}^{t} x(\tau) \, d\tau + y(0), \text{ for } t \geq 0 \]

a. Find impulse response \( h(t) \).
b. Is the system stable?

**Convolution**

Let, 
\[ \delta_{\Delta}(t) = \begin{cases} 
\frac{1}{\Delta}, & 0 \leq t \leq \Delta \\
0, & \text{otherwise} 
\end{cases} \]

then
\[ x(t) \approx \sum_{k=-\infty}^{\infty} x(k\Delta) \delta_{\Delta}(t-k\Delta) \Delta \]

Let \( \Delta \to 0 \), then
A continuous-time function can be broken up into a summation of shifted impulse functions. 

\[ x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta) \delta_{\Delta}(t-k\Delta) \Delta = \int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) d\tau \]

Suppose an LTI continuous system has the response \( h_{\Delta}(t) \) of impulse \( \delta_{\Delta}(t) \)

1. Time Invariant  
   \( \delta_{\Delta}(t-k\Delta) \rightarrow h_{\Delta}(t-k\Delta) \)

2. Homogeneity  
   \( x(k\Delta) \delta_{\Delta}(t-k\Delta) \rightarrow x(k\Delta)h_{\Delta}(t-k\Delta) \)

3. Additivity
4. Let $\Delta \to 0$, then

$$x(t) \rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau) d\tau = x(t)*h(t) \quad \text{(Convolution Integral)}$$

where "*" is the convolution symbol.

Alternative form: $y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau) d\tau$

**Example:**

$h(t) = Ae^{\alpha t}u(t), \ \alpha < 0,$

$x(t) = Be^{\beta t}u(t), \ \beta < 0.$

$$y(t) = \int_{-\infty}^{\infty} Ae^{\alpha \tau}u(\tau)Be^{\beta(t-\tau)}u(t-\tau) d\tau$$

$$= \int_{0}^{t} Ae^{\alpha \tau}Be^{\beta(t-\tau)} d\tau$$

$$= \frac{AB}{\alpha - \beta} \left[ e^{\alpha t} - e^{\beta t} \right]_{t \geq 0}.$$
b. Unit impulse response and exponential input signal

\[ h(t) = A e^{\alpha t} u(t) \]

\[ x(t) = B e^{\beta t} u(t) \]

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c. Unit impulse response and time-reversed input as functions of \( \tau \)

\[ h(\tau) = A e^{\alpha \tau} u(\tau) \]

\[ x(-\tau) = B e^{\beta \tau} u(-\tau) \]

FIGURE 2.10 *Analytical convolution*
Example:

**FIGURE 2.11** System data
Problem 2.4  \( h(t) = e^{-t}u(t), \ x(t) = tu(t) \), find \( y(1) \) by MATLAB.

Analytic solution: \( y(1) = \frac{1}{e} \)

Problem 2.5  Convolution with Impulse functions.
Sinusoidal Steady-State Response

What is the output when the input is an AC signal, that is, a sinusoidal signal? Since
\[ e^{j\omega t} = \cos \omega t + j \sin \omega t \]
let,
\[ x(t) = 2 \cos \omega t = e^{j\omega t} + e^{-j\omega t} \]
then,
\[
y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau
\]
\[ = \int_{-\infty}^{\infty} h(\tau)(e^{j\omega(t-\tau)} + e^{-j\omega(t-\tau)})\,d\tau
\]
\[ = \int_{-\infty}^{\infty} h(\tau)e^{j\omega(t-\tau)}\,d\tau + \int_{-\infty}^{\infty} h(\tau)e^{-j\omega(t-\tau)}\,d\tau
\]
\[ = e^{j\omega t} \int_{-\infty}^{\infty} h(\tau)e^{-j\omega \tau}\,d\tau + e^{-j\omega t} \int_{-\infty}^{\infty} h(\tau)e^{j\omega \tau}\,d\tau
\]
Let
\[ \int_{-\infty}^{\infty} h(\tau)e^{-j\omega \tau}\,d\tau = H(j\omega) = |H(j\omega)|e^{-j/2}H(j\omega), \]
then
Conclusion: in general, when a sinusoidal signal

\[ x(t) = A \cos(\omega t + \alpha), \quad -\infty < t < \infty \]

is applied to a stable LTI system, the output is

\[ y(t) = A |H(j\omega)| \cos(\omega t + \angle H(j\omega) + \alpha). \]

That is, the output is also a sinusoidal signal.

**Problem 2.6**

A highpass filter has impulse response

\[ h(t) = \delta(t) - 10e^{-10t}u(t). \]

If

\[ x(t) = 5 + 5\cos(10t), \quad -\infty < t < \infty \]

Find \( y(t) \).

First, compute \( H(j\omega) \).

\[
H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega \tau} d\tau
= \int_{-\infty}^{\infty} \left[ \delta(\tau) - 10e^{-10\tau}u(\tau) \right] e^{-j\omega \tau} d\tau
= 1 - 10 \int_0^\infty e^{-10\tau}e^{-j\omega \tau} d\tau
= 1 + \frac{10e^{-10\omega}e^{-j\omega}}{10+j\omega}
= 1 + \frac{10e^{-10\omega}}{10+j\omega}
= \frac{10 + 10e^{-10\omega}}{10+j\omega}
\]

The input \( x(t) \) has two frequency components

1. \( \omega = 0 \): \( H(j0) = 0 \), no output.
2. $\omega=10$: $H(j10)=0.707e^{j0.785}$, the output is $y(t)=3.53\cos(10t+0.785)$

Alternative Path to $H(j\omega)$

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{L} b_k \frac{d^k x(t)}{dt^k} \Rightarrow \sum_{k=0}^{N} a_k H(j\omega)(j\omega)^k e^{j\omega t} = \sum_{k=0}^{L} b_k (j\omega)^k e^{j\omega t}$$

$$\Rightarrow H(j\omega) = \frac{\sum_{k=0}^{L} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

**Problem 2.7**

A bandpass analog filter is described by

$$\ddot{\nu}(t)+2\dot{\nu}(t)+100\nu(t)=100\dot{x}(t)$$

where $\nu(t)$ is the output and $x(t)$ is input.

a. Determine the frequency response.

b. Find the output of $x(t)=10+10\cos(10t)+100\cos(100t)$

$$H(j\omega)=\frac{100j\omega}{(j\omega)^2+2j\omega+100}$$

$H(0)=0$, $H(j10)=50$, $H(j100)=1.01e^{-j1.55}$
\begin{align*}
\mathbf{y}(t) &= 500 \cos(10t) + 101 \cos(100t - 1.55) \\
\text{State-Space Model}
\end{align*}

If there is no derivative term in the input, that is,
\[\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = b_0 x(t)\]

we can define
\[v_1(t) = y(t), \quad v_2(t) = \frac{dy(t)}{dt}, \quad v_3(t) = \frac{d^2 y(t)}{dt^2}, \ldots, v_N(t) = \frac{d^{N-1} y(t)}{dt^{N-1}}.\]

Then we can turn the original equation to a set of first-order differential equations:
\begin{align*}
\frac{dv_1(t)}{dt} &= \frac{dy(t)}{dt} = v_2(t) \\
\frac{dv_2(t)}{dt} &= \frac{d^2 y(t)}{dt^2} = v_3(t) \\
&\vdots \\
\frac{dv_{N-1}(t)}{dt} &= \frac{d^{N-1} y(t)}{dt^{N-1}} = v_N(t) \\
\frac{dv_N(t)}{dt} &= \frac{d^N y(t)}{dt^N} = -a_0 v_1(t) - a_1 v_2(t) - \cdots - a_{N-1} v_{N-1}(t) + b_0 x(t)
\end{align*}

We call \( v_1(t), v_2(t), \ldots, v_N(t) \) state variables and the following state vector:
\[\mathbf{v}(t) = \begin{bmatrix} v_1(t) & v_2(t) & \ldots & v_N(t) \end{bmatrix}^T.\]

Then, the set of equations become in matrix form
A common definition for the state of a system is as follows: The state of a system is a minimum set of quantities \( v_1(t), v_2(t), \ldots, v_N(t) \) which if known at \( t=t_0 \) are uniquely determined for \( t \geq t_0 \) by specifying the inputs to the system for \( t \geq t_0 \).

Why?

\[
\frac{dv(t)}{dt} = A v(t) + B x(t)
\]

The outputs \( y(t) \) of a system are related to the states \( v(t) \) and a single input \( x(t) \) by the output equation

\[
y(t) = C v(t) + D x(t)
\]

where \( C \) is an \( M \) by \( N \) matrix and \( D \) is an \( M \) by 1 vector.

**Problem 2.8**

\[
\dot{\theta}(t) = \theta(t) + x(t), \quad \ddot{\theta}(t) = \beta \theta(t) - x(t)
\]

a. Describe the system in state variable form with \( v_1(t) = \theta(t) \) and \( v_3(t) = p(t) \).

b. Find the output equation if \( y_1(t) = \theta(t) \) and \( y_2(t) = p(t) \).
System Simulation (Numerical Solution Using MATLAB)

Consider

$$\ddot{\theta}(t) + 3\dot{\theta}(t) + 2\theta(t) = \frac{1}{3} \lambda(t)$$

Let

$$v_1(t) = \theta(t), \quad v_2(t) = \omega(t) = \dot{v}_1(t), \quad \lambda(t) = x(t),$$

then

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0.$$

Case 1: initial condition $v_1(0) = 1, \quad v_2(0) = 0. \quad x(t) = 0. \quad 0 \leq t \leq 5.$

A=[0 1; -2 -3];
B=[0;1/3];
C=[1 0];
D=0;
v0=[1;0];
tspan=[0 5];
x=@(t) 0;
df=@(t,v) A*v+B*x(t);
[t vvv]=ode45(df,tspan,v0);
plot(t,vvv(:,1));
xlabel('t');
ylabel('theta');

Case 2: initial condition $v_1(0) = 0, \quad v_2(0) = 0. \quad x(t) = 1. \quad 0 \leq t \leq 5.$
v0=[0;0];
x=@(t) 1;
df=@(t,v) A*v+B*x(t);
[t vv]=ode45(df,tspan,v0);
plot(t,vv(:,1));
xlabel('t');
ylabel('theta');

Example 2.1: An Oscillatory System

\[
\ddot{y}(t) + \omega_0^2 y(t) = kx(t)
\]

a. Find the characteristic equation and roots.
b. Solve for initial conditions of $y(0)=2$, $\dot{y}(0)=0$ and $x(t)=0$.
c. Write the state-space equation and use MATLAB to solve the equation. Assume $\omega_0=2\pi$. Plot the result with (b) to compare.
Example 2.2: Second-Order Systems

\[ \ddot{y}(t) + 2\zeta \omega_n \dot{y}(t) + \omega_n^2 y(t) = kx(t) \]

where \( \omega_n \) is the natural frequency.

a. From

\[ L \frac{di(t)}{dt} + Ri(t) + v(t) = e(t) \]
\[ C \frac{dv(t)}{dt} = i(t) \]

find

\[ \ddot{v}(t) + \alpha \dot{v}(t) + \beta v(t) = \gamma e(t) \]

b. Find the characteristic equation in terms of RLC circuit parameters and \((\omega_n, \zeta)\).

c. Let \( L \) and \( C \) fixed and \( 0 \leq R \leq \infty \), find the range of \( R \) that will yield

(i) Purely imaginary roots.

(ii) Complex roots.

(iii) Real roots.

\[ s_{1,2} = \frac{1}{2} \left\{ \frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}} \right\}, \quad \text{or} \quad s_{1,2} = \omega_n \left( -\zeta \pm \sqrt{\zeta^2 - 1} \right) \]

d. Find the state-space equation.

\[ \dot{\mathbf{v}}(t) = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta \omega_n \end{bmatrix} \mathbf{v}(t) + \begin{bmatrix} 0 \\ K \end{bmatrix} x(t) \]

e. Plot for the following cases: \((\omega_n = 10, \zeta = 0.707), (\omega_n = 10, \zeta = 0), (\omega_n = 10, \zeta = 2.3), (\omega_n = 10, \zeta = 0.1), (\omega_n = 100, \zeta = 0.1)\).

\[ \omega_n = 10; \]
\[ \zeta = 2.3; \]
\[ K = 100; \]
\[ A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta \omega_n \end{bmatrix}; \]
\[ B = [0; K]; \]
\[ v0 = [0; 0]; \]
\[ tspan = [0 2]; \]
\[ x = @(t) 1; \]
\[ df = @ (t, v) A*v + B*x(t); \]
\[ [t, vv] = \text{ode45}(df, tspan, v0); \]
\[ \text{plot}(t, vv(:, 1)); \]
\[ \text{xlabel('t');} \]
\[ \text{ylabel('theta');} \]

**FIGURE E2.2b**  
Locus of roots, \( 0 \leq R \leq \infty \)

**FIGURE E2.2c**  
Locus of roots, \( 0 \leq \zeta \leq \infty \)

**FIGURE E2.2d**  
Step response, \( \zeta = 0.707 \)  
*Comment:* It is rumored that pilots like this value of \( \zeta \), because it yields a rapid response with just a slight overshoot that won’t spill the coffee.

**FIGURE E2.2e**  
Step response, \( \zeta = 0 \)  
*Comment:* It is a fact that neither passengers nor crew like \( \zeta = 0 \).

**FIGURE E2.2f**  
Step response, \( \zeta = 2.3 \)  
*Comment:* They (passengers and crew) like this better, but it’s pretty slow.

**FIGURE E2.2g**  
Step response, \( \zeta = 0.1 \), natural frequency = 10
Example 2.3 Unit Impulse Response of a Lowpass Filter

A lowpass RC filter can be modeled by

\[ \dot{v}(t) + \frac{1}{RC} v(t) = \frac{1}{RC} e(t) \]

a. Find the characteristic equation.
b. Find \( v(t) \) for \( e_1(t) = \frac{1}{\Delta} u(t) \) and \( v(0) = 0 \).
c. Find \( v(t) \) for \( e_1(t) = \frac{1}{\Delta} u(t-\Delta) \) and \( v(0) = 0 \).
d. Find \( v(t) \) for \( e(t) = e_1(t) - e_2(t) \) and \( v(0) = 0 \).
e. Let \( \Delta \to 0 \), show that

\[ h(t) = \frac{1}{RC} e^{-t/\Delta} u(t) \]

delta=1;
A=-1;
B=1;
v0=0;
tspan=[0 4];
x=@(t) (t<=delta && t>=0)/delta;
df=@(t,v) A*v+B*x(t);
[t vv]=ode45(df,tspan,v0);
plot(t,vv);
xlabel('t');
ylabel('theta');
Example 2.4 Convolution

a. An LTI causal system is modeled by unit impulse response \( h(t) \).
Prove that for \( x(t)=u(t) \),
\[
\gamma(t) = \int_{0}^{t} h(\tau) d\tau
\]

b. A finite duration integrator can be modeled by the unit impulse response \( h(t)=u(t)-u(t-a) \). If the input is
\[ x(t) = Ae^{-bt}u(t), \quad b > 0, \]

Find the output by graphic method.

c. Find the output by analytical method.

d. If the hypothetical integrator is modeled by the noncausal unit impulse response \( h(t) = u(t + \frac{a}{2}) - u(t - \frac{a}{2}) \), find \( y(t) \).

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**Figure E2.4c** Plots of \( h(\tau) \) and a folded \( x(-\tau) \).

**Figure E2.4d** Integrand for \( 0 \leq t \leq a \)

**Figure E2.4e** Integrand for \( t > a \)

**Figure E2.4f** Approximation of output \( y(t) \)